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ELECTRIC TRANSMISSION LINES

Distributed Constants, Theory and Applications

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ELECTRIC TRANSMISSION LINES

Distributed Constants, Theory and Applications

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ELECTRIC TRANSMISSION LINES

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PREFACE

Twenty-five or thirty years ago, books on electric transmission lines were most commonly concerned with transmission of power at a single frequency. Some time later, reflecting the growth of long-distance telephone systems, authors gave attention to transmission of the audio-frequency band as well as power frequencies. More recently, radio-frequency transmission lines have come to be of great importance and value, and our practical interest now extends on through the radio-broadcast range and even to frequencies so high that transmission lines give place to hollow wave guides.

As electrical engineers have come to use higher frequencies, colleges have extended their courses of instruction. Some faculties still teach transmission-line material in each of two or three courses covering different parts of the frequency range. Others now present a single unified course that covers transmission lines at all frequencies.

This book is a presentation of the theory of circuits with distributed constants, which is valid at all frequencies, followed by chapters on the application of this theory to radio-frequency lines, power lines, telephone lines, filters, and wave guides. It is written primarily for the more general type of course that treats lines at all frequencies, although by appropriate selection of material (as will be explained shortly), it can also be used for a course relating to one type of application only. The advantages of the more general course need hardly be mentioned. If transmission lines are studied in several specialized courses, there is always some duplication of effort, a waste of the student's time. On the other hand, a program of study that teaches only one type of transmission-line application is in danger of overspecialization. The different aspects of transmission-line operation that appear in different parts of the frequency range recall the story of three blind men describing an elephant. Their ideas of the elephant seemed quite at variance—one feeling the trunk, one feeling the tail, one feeling the legs—and yet they were all investigating the same animal. Transmission lines are all one animal and should be seen as a whole.

Hence, the purpose of this book is to give as much on circuits with distributed constants, and their practical applications, as is needed by all electrical engineers. Where there is doubt, the book leans toward giving too much material rather than too little, but it is not intended to train the specialist. A man who devotes much of his time to telephone transmission will, of course, go beyond the scope of this book; he will not be satisfied with any one book, but will require a small library. The same is true of those who are largely engaged in the engineering of power lines or radio-frequency lines. Even the expert needs a breadth of knowledge of transmission lines beyond his own specialty, however, so this book is offered for him as well as for others.

The reader is expected to be well acquainted with ordinary circuit theory involving lumped constants and to have some competence in differentiation and integration. That is all. More advanced mathematics is sometimes mentioned, but only in footnotes that contain matters of interest lying a little apart from the main stream of the discussion.

When starting to write this book, I was faced with the question that must confront all authors in a changing field. Should the book present transmission-line material in ways that have become more or less traditional (and thereby retain the respect of instructors who are accustomed to that familiar organization) or should it be arranged to take advantage of modern developments? The question was answered by trying new ideas in classroom teaching. One or two deviations from custom proved well worth while.

For example, an easy approach to transmission-line theory is offered by the approximate formulas for high-frequency lines. The transmission-line equations, when applied to lines operating at high radio frequency, are greatly simplified and can be expressed in terms of the ordinary circular sine and cosine functions of real variables. If students are given these simple trigonometric forms immediately after an introductory discussion of traveling waves, and before the equations are generalized to complex hyperbolic form, the students' work is easier. Instead of having three new concepts (distributed constants, hyperbolic functions, and complex variables) to assimilate all at once, they can work with these ideas, which are strange to them, and a little difficult, one at a time. In this book, the idea of distributed constants comes first, then real circular functions, next come real hyperbolic functions, and only then are complex trigonometric or hyperbolic functions required.

Such an approach, by way of the high-frequency formulas, is specially appropriate in view of student interest, which seems to be most readily captured by ideas that relate to radio. Expression of the general transmission-line equations in terms of complex circular functions (the sine, cosine, and tangent with complex variables) naturally follows from the high-frequency presentation. Complex hyperbolic functions could be avoided entirely. However, because of their wide use in the literature, they are included, along with complex circular functions, throughout this book.

Hyperbolic functions of complex variables are a splendid development, historically. They are the natural form of mathematical expression for telegraph and cable lines. They describe low-frequency phenomena, and reduce to real hyperbolic functions for direct current. Their use in telegraph theory in the nineteenth century by such men as William Thomson, (Lord Kelvin) and Oliver Heaviside, showed great mathematical ingenuity. Today complex hyperbolic functions in transmission-line work are a little anachronistic. They do no harm except to puzzle the students at first, but considering the modern emphasis on high-frequency transmission rather than on direct-current telegraphy, circular trigonometric functions fit more naturally into the mathematics.

The plan of the book has been developed through ten years of teaching. The manuscript has gone through several editions of syllabus, rewritten as experience showed the need of revision.

The book is so designed that parts of it can be included, and parts omitted, at the discretion of the instructor. The first five chapters are basic theory and are necessary for all that follows. Chapters 6 and 7 contain material on inductance, capacitance, and resistance that may be familiar from other studies; if so, these chapters can be omitted although occasional reference is made to their contents in the latter part of the book. The artificial-line concepts of Chap. 8 are essential to the discussion of telephone and power lines in Chaps. 9 and 11.

Chapter 10, on filters, is preferably preceded by Chap. 9 on telephone lines, but apart from this the chapters on applications (from 11 on) are independent of each other. Thus, after Chap. 8, one may go to Chap. 9 and 10 on telephone lines and filters, or to Chap. 11 on power lines, or to Chaps. 12 and 13 on radio-frequency lines, impedance charts, and wave guides, or to Chap. 14 on traveling waves. Thus emphasis can be placed as desired, on telephone applications, or on power lines, or on radio-frequency lines.

The difference between telephone, power, and radio-frequency lines does not lie in the theory. The same basic equations are applicable over ranges of frequency, current, and voltage that are really worthy of consideration. It is not even the different assumptions or the different approximations that chiefly distinguish power from telephone, or telephone from radio-frequency engineering. It is a different way of thinking, a difference of concept. The author of a book like this must try to think as a power engineer when writing one chapter, as a telephone engineer for another chapter, and as a radio engineer for another. This is so hard to do that I have gone to the specialists for aid. I am most grateful to W. R. Johnson of the Pacific Gas and Electric Company; D. I. Cone, W. L. Carter, G. A. Furniss, and G. E. Hulstede of the Pacific Telephone and Telegraph Company; and F. E. Terman, K. R. Spangenberg, L. M. Field, and others of the Stanford faculty, for helping me avoid unfortunate errors of concept and emphasis or twists of idiom.

The aid of P.-K. Tien, who has assisted me in presenting the course and preparing the manuscript, is gratefully acknowledged.

It would not be adequate to give formal thanks to my wife, Hazel Dillon Skilling, for assistance; she has worked with me throughout the preparation of the manuscript and it is her book as well as mine.

The liberal policy of Stanford University in encouraging publication, as administered by President J. E. Wallace Sterling and Dean F. E. Terman, has been an essential factor in making this book possible.

HUGH HILDRETH SKILLING

PALO ALTO, CALIF.
March, 1951

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CHAPTER 1

THE BASIC EQUATIONS

1. Distributed Constants. In work with ordinary circuits, it is considered that the resistance of a branch of the circuit can all be concentrated in one *lump*, even though we know that as a matter of fact it is distributed throughout the length of quite a long piece of wire. The same simplifying assumption is commonly applied to inductance. It is convenient, and permissible, to *lump* series elements of resistance and inductance in this way if the same current flows in each elementary part of the circuit. Thus the arrangement of Fig. 1-1*b* is equivalent for purposes of computation to that of Fig. 1-1*a*, for current is the same in each of the elements of resistance and inductance of Fig. 1-1*a*.

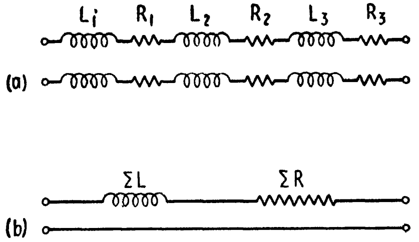

FIG. 1-1. Lumped series impedances.

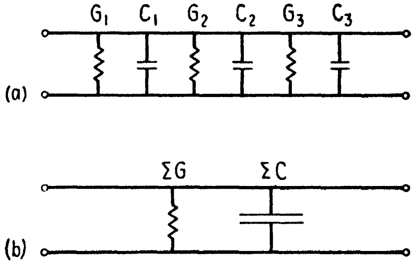

FIG. 1-2. Lumped shunt admittances.

In a somewhat similar manner, the conductances and capacitances of Fig. 1-2*a* can be combined as in Fig. 1-2*b*. Elements in parallel can be thus combined if the same voltage is applied across each of them.

Figure 1-3*a* is an illustration of a network that cannot be simplified by merely *lumping* the elements. The various branches carry different currents, and have different voltages across their terminals. Figure 1-3*b* shows a network in which a small amount of simplification is possible: the terminal currents are related in such a way that it is necessary for currents in the upper row of impedances to be equal to corresponding currents in the lower row of impedances; hence each impedance in the lower row can be combined with the impedance directly above it, giving a network similar to that in Fig. 1-3*a*.

A transmission line is comparable to the network of Fig. 1-3*b*. The conductors of the line have resistance and inductance in every unit length of the line. There is capacitance from conductor to conductor, and possibly there is conductance through leaky insulation, in every unit length of line.

Current and voltage both vary continuously along a transmission line. Current at one end of any short section of line differs from current at the other end because of the current that is shunted through the capacitance from wire to wire, and because of leakage current through the conductance of leaky insulation. Voltage at one end of any section of the line differs from voltage at the other end of the section because of inductance and resistance in the line. Indeed, the characteristic of a transmission line that distinguishes it from any other kind of a circuit is having both series and shunt parameters distributed along the circuit.

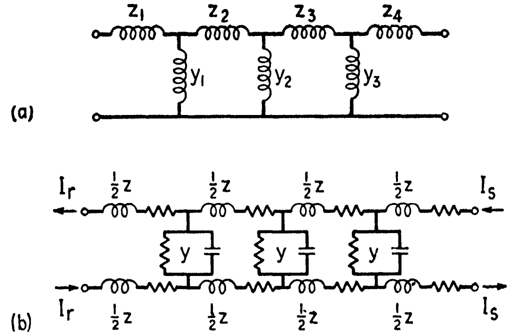

FIG. 1-3. Networks that cannot be lumped.

Not every pair of wires need be considered a transmission line. The mathematics of transmission lines should be applied to a pair of wires if both series and shunt parameters are important. This will always be true on a line so long that its length is a considerable fraction of the transmitted wavelength; that is, if the frequency is high or the line long. The same treatment is needed, however, even at low frequency, if wire resistance and insulation leakage are both rather large (as may be true, for instance, in a cable).

Since an "electrically long" line is one with a length of the order of magnitude of a wavelength, or at least a reasonable fraction of a wavelength, an inch-long line ($2\frac{1}{2}$ centimeters) is electrically "long" for a 3,000-megacycle current with 10-centimeter wavelength. A telephone line of a few miles is "long" for a current of voice frequency. A power line carrying 60-cycle current is hardly to be considered electrically "long" unless its actual length is a hundred miles or more.

2. Types of Lines. A line of two parallel wires is a common form of transmission line. A telephone line is an example. Two-wire lines are also used for radio-frequency transmission, but only at fairly low radio frequencies; a two-wire line might be used, for instance, between a transmitter operating in the standard broadcast frequency band and its antenna.

For high radio-frequency work it is usually better to use a *coaxial* line; one conductor is a copper wire (or small tube) running inside a larger copper tube which serves as the second conductor. The inner conductor is supported at the center of the enclosing tube by insulating material. If the coaxial line is rigid, the inner conductor may be supported by insulating *beads* strung along it at intervals, but if the line is flexible the outer tube is commonly of copper braid, and pliant insulation fills the space between conductors.

A three-phase power line is not essentially different from a single-phase line. Most practical power lines can be handled as readily as two-wire lines by a simple artifice that will be introduced later.

The basic mathematics of transmission lines applies to lines of all configurations, and for the present it will perhaps be easiest to visualize the familiar two-wire line. The differential equations, however, and their solutions, are equally applicable to coaxial, polyphase, and multiconductor lines.

3. The Differential Equations. The fundamental differential equations of transmission-line operation are derived from experimental evidence. By observation and measurement it is found that voltage varies from point to point along a line and that current in the line also changes from one point to another.

Line Resistance and Inductance. Consider a transmission line carrying alternating (sine-wave) current. Voltage is measured at each end of a section of line. The voltage at the receiving end is different from the voltage at the sending end, and subtracting the former from the latter gives the voltage drop in the length of the line under consideration:

$$\Delta V = V_s - V_r \tag{1-1}$$

The voltage difference thus obtained is then divided by the length of the section of line, which may be called Δx, giving

$$\frac{\Delta V}{\Delta x} \tag{1-2}$$

This ratio is a kind of average voltage drop per unit length of line in the section Δx. If Δx approaches zero, the ratio approaches a limiting value; the limit of $\Delta V/\Delta x$ is dV/dx, the rate of change of voltage with respect to distance at a specific point on the line.

It is natural to expect, from previous knowledge of circuits, that the voltage drop along the line should be proportional to current in the line, and experiment shows this to be true. The rate of change of voltage with respect to distance, dV/dx, is found to be equal to the current times a constant, and we write

$$\frac{dV}{dx} = zI \tag{1-3}$$

where I is the current at the point of the line under consideration and z is a constant. Since dV/dx is the voltage drop per unit length of line, we call z the impedance per unit length of line.

In general, the voltage drop is not in phase with the current; so the impedance z is a complex quantity. One component of the voltage drop is in phase with the current, and another component is in quadrature with the current. This relation is taken into account by letting z have real and imaginary components of appropriate value; the real component is called resistance per unit length of line, and the imaginary component is called reactance per unit length.

The line *resistance*, defined in this way, is, of course, equal to the ordinary d-c resistance of the two wires of the line if the frequency is low. Resistance of conductors of standard size can be found in the wire tables of handbooks; note that for a two-wire line the resistance per mile of line is twice the resistance per mile of wire. At high frequency, however, the value of resistance is increased by *skin effect*, which results from nonuniform distribution of current density in the conductor. Skin effect will be discussed in Chap. 7.

Line *reactance* results from the magnetic field that is produced between and around the line conductors by the current. As we should expect, measurement shows that the reactance of a line is proportional to the frequency f of the current.

To obtain a constant that is independent of frequency, it is convenient to express the reactance in terms of *inductance*, l, inductance being so defined that

$$\text{Reactance} = 2\pi fl \tag{1-4}$$

Inductance is substantially independent of frequency. Indeed, this value of inductance is the same as the value that can be computed from the magnetic field about a transmission line carrying steady direct current.[1]

---

[1] Provided resistance is not excessive. It should be emphasized that when line inductance is multiplied by $2\pi f$ times the current flowing in the wire, the result is the magnitude of reactive drop along that wire. This we take as the definition of inductance. Use of this concept often avoids later misunderstanding. Although

The formulas used for computation of inductance of a two-wire line are numbered 1 and 3 in Table I (inside front cover), and Formula 7 is for a coaxial line.

Introducing resistance r and inductance l into Eq. 1-3,

$$\frac{dV}{dx} = zI = (r + j2\pi fl)I \tag{1-5}$$

If the usual abbreviation of ω for $2\pi f$ is used,

$$\frac{dV}{dx} = zI = (r + j\omega l)I \tag{1-6}$$

This is one of the two differential equations of the transmission line for steady-state operation.[2] V, I, and z are, in general, complex quantities.

---

inductance is computed from a magnetic field that is assumed to have a certain configuration about the conductor, for both practical and theoretical reasons it is better to *define* inductance in terms of voltage drop. The practical reason is that voltage is the measurable quantity, the quantity that affects line operation, the quantity of practical importance, *the observed quantity from which inductance is computed.* The theoretical reason is that there is a lack of rigor in assuming that the electric and magnetic field configurations determined experimentally for direct current are applicable to high-frequency transmission until this is proved (or, in a line with appreciable losses, is shown to be a good approximation) by means of electromagnetic field theory. (See, for instance, H. H. Skilling, "Fundamentals of Electric Waves," Ref. 29 in list of references, Appendix 4.)

[2] The usual conventions of complex-quantity notation are used here. V and I are complex and represent magnitude and relative phase of sinusoidal, steady-state voltage and current. More generally, letting v and i be instantaneous values, instead of Eq. 1-6 we have

$$\frac{\partial v}{\partial x} = ri + l\frac{\partial i}{\partial t} \tag{1-7}$$

Assuming i sinusoidal, and using the symbol Re to indicate "Real part of":

$$i = \text{Re}\,\{I_{\max}e^{j(\omega t+\theta)}\}$$

The differential equation being linear, v is also sinusoidal:

$$v = \text{Re}\,\{V_{\max}e^{j(\omega t+\phi)}\}$$

Substitution of these expressions into Eq. 1-7 gives

$$\frac{\partial}{\partial x}\text{Re}\,\{V_{\max}e^{j(\omega t+\phi)}\} = r\,\text{Re}\,\{I_{\max}e^{j(\omega t+\theta)}\} + l\frac{\partial}{\partial t}\text{Re}\,\{I_{\max}e^{j(\omega t+\theta)}\} \tag{1-8}$$

$$= r\,\text{Re}\,\{I_{\max}e^{j(\omega t+\theta)}\} + l\,\text{Re}\,\{j\omega I_{\max}e^{j(\omega t+\theta)}\} \tag{1-9}$$

$$= \text{Re}\,\{(r + j\omega l)I_{\max}e^{j(\omega t+\theta)}\} \tag{1-10}$$

Equation 1-9 is obtained from the derivative of an exponential as expanded by Eu-

If impedance, resistance, and reactance are in ohms per meter (or per mile), V and I are in volts and amperes.

Line Conductance and Capacitance. If current is measured at each end of a section of line, the two measurements are not, in general, the same. Subtracting the current at the receiving end of the section from that at the sending end gives the difference:

$$\Delta I = I_s - I_r \tag{1-13}$$

The average change of voltage per unit length of line is found by dividing by the length of the section Δx, giving $\Delta I/\Delta x$. If the length of line under consideration is shortened so that Δx approaches zero, the limit of the ratio is dI/dx. This is the rate of change of current with respect to distance along the line at a particular point of the line.

It has already been mentioned that there are two reasons for the observed difference between I_s and I_r. One is the current that leaks from wire to wire of the line through imperfect insulation. The other is the charging current to or from the capacitance of the section of line under consideration as the voltage changes with time.

Let us assume that the line is carrying alternating (sine-wave) current, and the voltage is sinusoidal, and the usual conventions of complex-quantity notation are to be used. Experiment shows, as we should expect, that ΔI in any given section of line is proportional to the voltage from wire to wire, and the derivative can hence be written

$$\frac{dI}{dx} = yV \tag{1-14}$$

The constant y is called the admittance per unit length of the line.

Since dI/dx is not, in general, in phase with the line voltage, admittance[3] is a complex quantity.

---

ler's formula, and 1-10 from the rule for addition of complex quantities. For Eq. 1-10, it is sufficient that

$$\frac{\partial}{\partial x}\left[V_{\max}e^{j(\omega t+\phi)}\right] = (r + j\omega l)I_{\max}e^{j(\omega t+\theta)} \tag{1-11}$$

Dividing by $e^{j\omega t}$, and letting a complex quantity represent rms amplitude and phase in the customary manner by setting $V_{\max}e^{j\phi} = \sqrt{2}\,V$ and $I_{\max}e^{j\theta} = \sqrt{2}\,I$, there results

$$\frac{\partial}{\partial x}V = (r + j\omega l)I \tag{1-12}$$

This development of Eq. 1-6 shows the relation of the sinusoidal equation to the more general equation 1-7.

[3] It may be emphasized at once, to avoid possible confusion, that there is no

The real component of admittance (which corresponds to the component of incremental current that is in phase with the line voltage) is called the *conductance* per unit length, g. If the insulation of a line were so poor that current could leak from wire to wire, the line would have an appreciable conductance. As a matter of fact, it is unusual for insulation to be so poor that there is much leakage through the insulation, but other losses may contribute to an "effective" value of g, as will be discussed in Chap. 7.

The imaginary component of admittance is called *susceptance.* Susceptance accounts for the charging current to the distributed capacitance of the line, and experiment shows that susceptance is proportional to frequency. To obtain a constant that is independent of frequency, it is convenient to introduce *capacitance* per unit length, c. To define c,

$$\text{Susceptance} = 2\pi fc = \omega c \tag{1-15}$$

We find that c is the ordinary electrostatic capacitance per unit length of the line, and it can be computed from the electrostatic field.[4] Formulas 4 and 6 of Table I give the capacitance of a two-wire line; and Formula 8 is for a coaxial line.

Introducing conductance g and capacitance c into Eq. 1-14,

$$\frac{dI}{dx} = yV = (g + j\omega c)V \tag{1-16}$$

This equation and Eq. 1-6 are the two basic differential equations of the transmission line.[5]

4. Representation of Transmission-line Constants by Lumped Elements. If it is accepted that the distributed inductance, resistance, capacitance, and conductance of a transmission line can be represented by a series of small lumped elements, as in Fig. 1-3*b*, a different approach to the differential equations is available. This approach is quite frequently used (see, for instance, Refs. 7 and 30, Appendix 4). The validity of the concept is not self-evident, but it will be demonstrated in Chap. 8. This concept of lumped elements is helpful in visualizing transmission-line constants.

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reciprocal relation between z and y for a transmission line. They refer to different parts of the system as do the z and y indicated in Fig. 1-3*b*. Speaking somewhat crudely, z relates to the conductor and y to the insulation of the line.

[4] Comments in footnote 1 about inductance and its definition are applicable to the definition of capacitance, also. Capacitance is here defined in terms of charging current.

[5] The more general form of Eq. 1-16 comparable to Eq. 1-7, is

$$\frac{\partial i}{\partial x} = gv + c\frac{\partial v}{\partial t} \tag{1-17}$$

A transmission line can be approximately represented by a number of impedance and admittance elements in a ladder network, as in Fig. 1-3*b*. The representation is approximate if the elements are of finite size. If, however, the number of the elements is permitted to increase without limit, each individual element at the same time becoming infinitesimally small, the representation becomes exact.

Figure 1-4 shows the impedance of a section of line of infinitesimal length represented by a lump of inductance and a lump of resistance. Voltage at the end nearer the load is called V, and voltage at the farther end of the differential section is $V + dV$. Distance along the line, x, is measured *from* the load toward the sending end of the line. Knowing the resistance of the line to be r per unit length (ohms per meter), the resistance of this short section of length dx is $r\,dx$. Similarly, the inductance of this short section is $l\,dx$. The change of voltage in the section dx is

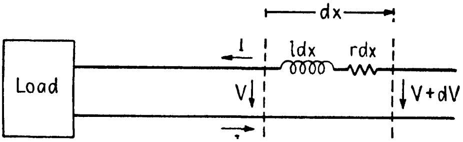

FIG. 1-4. The series elements of a transmission line.

$$dV = (r\,dx + j\omega l\,dx)I \tag{1-18}$$

This is in agreement with Eq. 1-6.

Consider next the change of current in this section of infinitesimal length. Call the current at one end of the section I, and that at the other end $I + dI$, as in Fig. 1-5. The capacitance of this short section is $c\,dx$, and the conductance is $g\,dx$. Current through the admittance of the section is

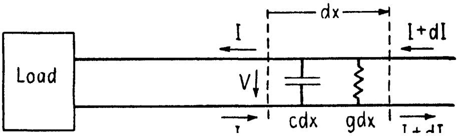

FIG. 1-5. The shunt elements of a transmission line.

$$dI = (g\,dx + j\omega c\,dx)V \tag{1-19}$$

as in Eq. 1-16.

It will be observed that lumps of capacitance and conductance were not included in Fig. 1-4, although they are known to exist. A truly rigorous development would take them into account, and they would cause current at one end of the section of differential length to differ from current at the other end. However, in the consideration of voltage, they add to Eq. 1-18 only second-order infinitesimal terms. Such second-order infinitesimal terms are discarded at the conclusion of the rigorous development, and Eq. 1-18 is the precise result. In a similar manner, inductance and resistance are omitted from the consideration of current in Fig. 1-5; they would, if included, produce only second-order infinitesimal terms in Eq. 1-19. Equations 1-18 and 1-19 are therefore correct.

In this way, the concept of a line consisting of infinitesimal lumps of inductance, resistance, capacitance, and conductance can be used to develop the differential equations.

5. Solution of the Differential Equations. The experimental relations of current and voltage at any point on a transmission line have been expressed as a pair of equations:

$$\frac{dV}{dx} = zI \tag{1-20}$$

$$\frac{dI}{dx} = yV \tag{1-21}$$

These equations are simple because the action of a transmission line is inherently simple, and also because certain assumptions and restrictions have been made in the derivation. The more important restrictions are:

1. Currents and voltages are steady-state and sinusoidal.
2. Line constants r, l, c, and g have values that are independent of current or voltage, and are not changing with time. It has not been assumed in the derivation to this point, but it will be assumed hereafter in the solution, that the line constants are uniform along the line, and are not functions of x.
3. Resistance is not excessive (see footnote 11, Chap. 6).

A solution of this pair of simultaneous differential equations is needed. A solution will be an expression for voltage and another for current, as functions of distance along the line, so that the voltage or current at any point may be computed for any given value of the distance x.

To obtain a solution, one variable must be eliminated from Eqs. 1-20 and 1-21. To eliminate I, Eq. 1-20 may be differentiated, giving

$$\frac{d^2V}{dx^2} = z\frac{dI}{dx} \tag{1-22}$$

The derivative of I which appears on the right-hand side of this equation is known from Eq. 1-21, and substitution gives

$$\frac{d^2V}{dx^2} = zyV \tag{1-23}$$

We now have an equation entirely in V. It is a second-order differential equation. It is necessary to find an expression for V that will satisfy this equation.

6. The Exponential Form of Solution. One possible solution of Eq. 1-23 is

$$V = V_1 e^{\sqrt{zy}\,x} \tag{1-24}$$

To show that this is a solution it may be substituted into the equation, which it satisfies. The value of V_1 can be chosen arbitrarily, for Eq. 1-24 is a solution of 1-23 regardless of the value of V_1.

Another solution of Eq. 1-23 is

$$V = V_2 e^{-\sqrt{zy}\,x} \tag{1-25}$$

This also will satisfy Eq. 1-23 when substituted for V, and V_2 may have any value. Furthermore, it is readily verified that the sum of these two expressions

$$V = V_1 e^{\sqrt{zy}\,x} + V_2 e^{-\sqrt{zy}\,x} \tag{1-26}$$

will also satisfy Eq. 1-23, and since this sum contains two arbitrary constants, it is the complete solution of the second-order differential Eq. 1-23.

The solution for current is exactly similar. To find current, V instead of I is eliminated from Eqs. 1-22 and 1-23, and the solution of the resulting second-order differential equation is

$$I = I_1 e^{\sqrt{zy}\,x} + I_2 e^{-\sqrt{zy}\,x} \tag{1-27}$$

Equations 1-26 and 1-27 describe voltage and current distributions along a transmission line. They satisfy the requirements of Eqs. 1-22 and 1-23 in every infinitesimal section of the line. They give the solution of the transmission-line differential equations in exponential form, and will lead to the concept of traveling waves.

Let us consider the meaning of one of the exponential terms, such as

$$V = V_1 e^{\sqrt{zy}\,x} \tag{1-28}$$

This represents a sinusoidal voltage[6] at a point on a transmission line. It is the voltage at a point that is at a distance x from the load end of the line.

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[6] Equation 1-28 represents a sinusoidal alternating voltage by the usual convention of vector or complex-quantity notation, the instantaneous voltage being

$$v = \mathrm{Re}\,\{\sqrt{2}\,V_1 e^{\sqrt{zy}\,x} e^{j\omega t}\} \tag{1-29}$$

See footnote 2.

If $x = 0$, the point in question is just at the load end of the line. If $x = 0$, $e^{\sqrt{zy}\,x} = 1$, and $V_1e^{\sqrt{zy}\,x}$ reduces to V_1; hence V_1 is the voltage represented by this term at the load end of the line.

At any other point $V_1e^{\sqrt{zy}\,x}$ differs from V_1. It differs in both magnitude and angle, for z and y are in general complex and the exponential is therefore a complex quantity. This means that V of Eq. 1-28 represents different voltages at different points along the line, voltages that differ in both amplitude and phase.

An example is shown in Fig. 1-6. The voltage vectors of Fig. 1-6 illustrate values of $V_1e^{\sqrt{zy}\,x}$ when the values of z and y are typical of a telephone line at a frequency of 1,000 cycles. As x, the distance, increases, the voltage represented by $V_1e^{\sqrt{zy}\,x}$ becomes larger in magnitude and more advanced in angle.

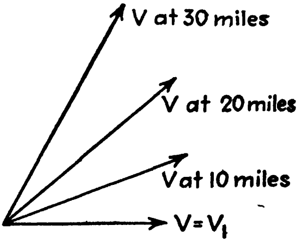

FIG. 1-6. Relative voltages on a typical telephone line.

In general, Eq. 1-28 is the expression for only one component of voltage on a transmission line. The complete expression, as in Eq. 1-26, contains two exponential terms. The voltage at any point of the line is therefore the vector sum of two terms of the kind we have been considering.

It was shown that the value of $V_1e^{\sqrt{zy}\,x}$ at the load end of the line is merely V_1. Similarly, when both exponential terms of Eq. 1-26 are taken into account, the receiving-end voltage is seen to be

$$V_r = V_1 + V_2 \tag{1-30}$$

The receiving-end voltage is indicated by the subscript r. From Eq. 1-27, the similar expression for receiving-end current is

$$I_r = I_1 + I_2 \tag{1-31}$$

In general, V_1 and V_2 and I_1 and I_2 are all complex quantities.

Characteristic Impedance and Other Parameters. Equations 1-26 and 1-27 describe voltage and current on a transmission line in terms of the voltage components V_1 and V_2 and the current components I_1 and I_2, but there is a necessary relation between voltage and current that is not included in these equations. This relation was implicit in the differential equations 1-20 and 1-21, but was lost in the simultaneous solution. It may be recovered by substituting Eqs. 1-26 and 1-27 into 1-20 and 1-21, giving

$$\sqrt{zy}\,V_1e^{\sqrt{zy}\,x} - \sqrt{zy}\,V_2e^{-\sqrt{zy}\,x} = zI_1e^{\sqrt{zy}\,x} + zI_2e^{-\sqrt{zy}\,x} \tag{1-32}$$

$$\sqrt{zy}\,I_1e^{\sqrt{zy}\,x} - \sqrt{zy}\,I_2e^{-\sqrt{zy}\,x} = yV_1e^{\sqrt{zy}\,x} + yV_2e^{-\sqrt{zy}\,x} \tag{1-33}$$

Divide Eq. 1-32 by $\sqrt{zy}$, and Eq. 1-33 by y, obtaining

$$V_1 e^{\sqrt{zy}\,x} - V_2 e^{-\sqrt{zy}\,x} = \sqrt{\frac{z}{y}}\, I_1 e^{\sqrt{zy}\,x} + \sqrt{\frac{z}{y}}\, I_2 e^{-\sqrt{zy}\,x} \tag{1-34}$$

$$V_1 e^{\sqrt{zy}\,x} + V_2 e^{-\sqrt{zy}\,x} = \sqrt{\frac{z}{y}}\, I_1 e^{\sqrt{zy}\,x} - \sqrt{\frac{z}{y}}\, I_2 e^{-\sqrt{zy}\,x} \tag{1-35}$$

Adding Eqs. 1-34 and 1-35, it is found that

$$V_1 = \sqrt{\frac{z}{y}}\, I_1 \tag{1-36}$$

and by subtracting Eq. 1-34 from Eq. 1-35

$$V_2 = -\sqrt{\frac{z}{y}}\, I_2 \tag{1-37}$$

Equations 1-36 and 1-37 show that current and voltage are related by the quantity $\sqrt{z/y}$, with positive sign for one pair of components and negative sign for the other pair.

This quantity, $\sqrt{z/y}$, is used so often that it is given a special name and a special symbol: it is called *characteristic impedance* and is written Z_0. Thus

$$Z_0 = \sqrt{\frac{z}{y}} \tag{1-38}$$

Using this abbreviation, Eqs. 1-36 and 1-37 become $V_1 = Z_0 I_1$ and $V_2 = -Z_0 I_2$. Certain other quantities also have names and special symbols. Thus $\sqrt{zy}$ is called the *propagation constant* and is often represented by γ:

$$\gamma = \sqrt{zy} \tag{1-39}$$

Since both z and y are in general complex quantities, the propagation constant will generally be complex. It will have a real part and an imaginary part, and for convenience these are abbreviated as α and $j\beta$:

$$\gamma = \sqrt{zy} = \alpha + j\beta \tag{1-40}$$

The real part α is called the *attenuation constant* for reasons that will appear later, and β is called the *phase constant.*

Introducing these new symbols into Eqs. 1-26 and 1-27,

$$V = V_1 e^{\alpha x} e^{j\beta x} + V_2 e^{-\alpha x} e^{-j\beta x} \tag{1-41}$$

$$= V_1 e^{\alpha x} \underline{/\beta x} + V_2 e^{-\alpha x} \underline{/-\beta x} \tag{1-42}$$

$$I = I_1 e^{\alpha x} e^{j\beta x} + I_2 e^{-\alpha x} e^{-j\beta x} \tag{1-43}$$

$$= I_1 e^{\alpha x} \underline{/\beta x} + I_2 e^{-\alpha x} \underline{/-\beta x} \tag{1-44}$$

These equations are written in two notations. The complex form $Ae^{j\beta x}$ may be more familiar to some readers and the form $A\ \underline{/\beta x}$ to others.[7]

7. Traveling Waves. Let us again inquire into the physical meaning of such a term as $V_1e^{\sqrt{zy}\,x}$ or, as we shall now prefer to write, $V_1e^{\alpha x}e^{j\beta x}$. It was previously mentioned, and illustrated in Fig. 1-6, that this represents an alternating-voltage vector, the magnitude and angle of which depend on the value of x. It is now seen that the change in *magnitude* for various values of x depends on the term $e^{\alpha x}$ alone, whereas the change of *phase* from one point on the line to another is given solely by $e^{j\beta x}$.

Thus in Fig. 1-6 the voltage vectors are increasingly longer as distance from the load end of the line is increased, and this is due to the increasing value of $e^{\alpha x}$; the increase in the length of the vector being exponential with x. At the same time, the voltage vector is advanced in phase as distance from the load increases, owing to the increasing angle of $e^{j\beta x}$. The increase of angle is directly proportional to x.

These relations suggest a concept of the electrical behavior of a line that turns out to be extraordinarily useful. Consider that an a-c generator is producing sinusoidal voltage at the sending end of the line and that the voltage is propagated along the line, toward the load, as a traveling wave. (The analogy of holding a rope taut and shaking one end rapidly up and down to produce a traveling wave is obvious.) As the traveling wave passes any point on the line it appears, at that point, as an alternating voltage.

If the assumed wave has finite velocity, it requires time to travel. The traveling wave requires time to reach various points along the line with a delay that is in proportion to their distance from the generator. The traveling wave produces an alternating voltage at each point as it passes. Since the wave produces voltage at a slightly later instant at a more distant point, the alternating voltage at the more distant point appears to be relatively lagging in phase. If the wave travels with constant velocity, the angle of lag of the vector that represents voltage at any particular point is proportional to the distance of that point from the generator.

If this same assumed traveling wave becomes smaller as it travels, as a result of losing energy, the amplitude of the alternating voltage at a point near the load is less than the amplitude at a point near the generator. Combining this change of amplitude with the change of phase: *as a wave travels along the line toward the load, voltage diminishes and phase becomes more lagging.*

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[7] Equivalence is shown from Euler's formula: $e^{j\beta x} = \cos\beta x + j\sin\beta x$. Plotting the right-hand side of this equation in the complex plane, the sum is a vector of unit length at an angle βx. Hence $e^{j\beta x}$ is a complex quantity of magnitude one and angle βx.

This sentence describes the appearance of a traveling wave. However, this sentence also describes the appearance of the voltage defined by the exponential term $V_1e^{\alpha x}e^{j\beta x}$, as was shown above. Therefore this exponential term may be interpreted as the mathematical description of a traveling wave.[8]

If $V_1e^{\alpha x}e^{j\beta x}$ is a wave traveling from the generator toward the load, $V_2e^{-\alpha x}e^{-j\beta x}$ is a wave traveling in the opposite direction, from load toward generator. The two expressions differ only in the sign associated with x, and hence in the sign of the velocity. The first expression is for a wave that grows smaller and more lagging in phase as it travels *toward* the load. The second is for a wave that grows smaller and more lagging as it travels *from* the load.

Looking back to Eq. 1-41, it is seen that the voltage on a transmission line can be described as the sum of two traveling waves, one traveling toward the load and the other from the load. The first is called the *incident wave,* the second the *reflected wave.*

Current, as in Eq. 1-43, is also the sum of two traveling waves, an incident wave and a reflected wave. From Eq. 1-36 it is known that the voltage of the incident wave and the current of the incident wave are related by the characteristic impedance

$$V_1 = Z_0I_1 \tag{1-46}$$

From Eq. 1-37, the voltage and current of the reflected wave are also related by the characteristic impedance but with negative[9] sign.

$$V_2 = -Z_0I_2 \tag{1-47}$$

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[8] Instantaneous voltage is

$$\begin{aligned} v &= \sqrt{2}\,V_1e^{\alpha x}e^{j\beta x}e^{j\omega t} \\ &= \sqrt{2}\,V_1e^{\alpha x}e^{j(\beta x+\omega t)} \\ &= \sqrt{2}\,V_1e^{\alpha x}e^{j\beta[x+(\omega/\beta)t]} \end{aligned} \tag{1-45}$$

When thus fully written out, the exponential term for voltage is seen to have the usual mathematical form for a traveling wave: it is a function of $x + (\omega/\beta)t$, and this indicates that it has a velocity of ω/β in the negative x direction and is therefore moving from the generator toward the load.

[9] Whether the negative sign is associated with the incident wave or the reflected wave is a matter of convention. In Fig. 1-4 the positive directions of current and voltage are assumed as shown. Positive power is therefore toward the left, and for this reason the left end of the line is designated the load end. Consistent with this assumption, a wave of positive voltage that is conveying energy toward the load will be accompanied by a current in the direction that is called *positive*; whereas a wave of positive voltage that is conveying energy away from the load will be accompanied by a *negative* current. A negative current merely means a current opposite to the arrows in the figure.

The electrical behavior of a transmission line can thus be described in terms of two traveling waves. The incident wave travels from the generator to the load. Usually part of this wave is sent back from the load toward the generator, constituting the reflected wave. There are both voltage and current components in each traveling wave, and they are related by the line's characteristic impedance Z_0.

Total voltage and current at any point of the line are found by adding the voltages and currents of the two waves at that point. For instance, voltage at the load is the sum of the voltage of the incident wave at the load and the voltage of the reflected wave at the load. Voltage at the generator is the sum of the voltages of the incident and reflected waves at the generator. Note that the generator voltage is not the voltage of the incident wave only, for the reflected wave also contributes to voltage (and to current) at the generator, as it does everywhere else along the line.

Wavelength and Velocity. If there are waves on a transmission line, they must have a velocity of propagation and a wavelength. The wavelength is defined as the distance between points at which the alternating voltages (or currents) produced by the wave differ in phase by one complete cycle or 2π radians. It is therefore the value of x, the distance, that makes βx, the angle of the voltage vector, equal 2π. The wavelength is called λ, and since λ is so defined that $\beta x = 2\pi$ when $x = \lambda$, it follows that $\beta\lambda = 2\pi$ and

$$\lambda = \frac{2\pi}{\beta} \tag{1-48}$$

Since β is known from the line constants, λ is easily found.

A wave travels 1 wavelength in the time of 1 cycle. It travels as many wavelengths in a second as there are cycles in a second. Hence

$$\text{Velocity} = \lambda f = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} \tag{1-49}$$

This is the velocity of phase propagation, or phase velocity. It is the speed at which waves would appear to travel if the electric and magnetic fields were visible (see also footnote 8 regarding velocity).

It is entirely possible to account for all transmission-line action in terms of traveling waves, but computation is often easier if other mathematical functions are used instead of the exponential functions that correspond to the traveling-wave concept. Lines with low losses, for example, are especially easily handled with trigonometric functions, as appears in Chap. 3.

PROBLEMS

1-1. Give the lengths of the following lines in wavelengths, assuming velocity of propagation is equal to the speed of light.

a. A telephone line from Albany to Troy (7 miles) at 1,000 cycles.
b. A telephone line from Albany to Ithaca (150 miles) at 1,000 cycles.
c. A power line from Jordan Dam to Meridian, Miss. (150 miles) at 60 cycles.
d. A broadcast transmitter line, 75 meters long, at 1 megacycle.
e. A line 1 foot long at 10,000 megacycles.

1-2. Give the lengths of the following lines in wavelengths, assuming velocity of propagation is equal to the speed of light.

a. A telephone line from Springfield to Decatur (35 miles) at 1,000 cycles.
b. A telephone line from Springfield to Riverton (6 miles) at 1,000 cycles.
c. A power line from Grace, Idaho, to Salt Lake City (135 miles) at 60 cycles.
d. A broadcast transmitter line, 115 meters long, at 800 kilocycles.
e. A line 8 centimeters long at 3,000 megacycles.

1-3. Equation 1-23 is the voltage-wave equation. Derive a similar current-wave equation.

1-4. Show by direct substitution that Eq. 1-26 will satisfy Eq. 1-23.

1-5. Show by direct substitution that Eq. 1-27 will satisfy the current equation derived in Prob. 1-3.

1-6. The phase constant of a telephone line is 0.036 (radians) per mile and the attenuation constant is 0.0078 per mile. Find values of voltage of a traveling wave at distances of 15, 30, and 45 miles, and draw vectors as in Fig. 1-6.

1-7. The velocity of propagation on a telephone line is 179,000 miles per second, and the attenuation constant is 0.0036 per mile. Find values of voltage of a traveling wave at distances of 20, 40, and 60 miles, and draw vectors as in Fig. 1-6. (The line of Fig. 1-6 has smaller wires.)

1-8. Values in Prob. 1-6 are at 1,000 cycles per second. Repeat the problem for a frequency of 500 cycles. To do this, you must assume that certain quantities do not vary with frequency; tell which. Explain other assumptions that you find necessary.

1-9. A telephone line consists of two No. 10 AWG copper wires (102 mils diameter) spaced 10 inches apart. (*a*) Find l and c per mile from the formulas of Table I, using $K_\varphi = 1$. (*b*) Find z and y at 1,000 cycles, assuming $g = 0$. (*c*) Find Z_0, α, and β.

Note: Do not confuse radius and diameter when finding l and c; this is a common mistake.

1-10. Repeat Prob. 1-9 for a telephone line of No. 8 AWG copper wires (128.5 mils diameter).

1-11. A television lead-in line consists of two No. 10 copper wires (102 mils diameter) spaced 0.65 inch apart. Assuming $K_\varphi = 0$ and neglecting r and g, find Z_0 and β per meter at 200 megacycles.

1-12. A coaxial line consists of a wire of 1.30 millimeters diameter within a tube the inner diameter of which is 5.50 millimeters, the space between conductors being filled with air. Find Z_0 and β at 1,000 megacycles, assuming $K_\varphi = 0$ and neglecting r and g.

1-13. For the line of Prob. 1-9, compute and plot l and c as functions of the spacing between wires (from 1 to 60 inches).

1-14. For the line of Prob. 1-9, compute and plot l and c as functions of the wire diameter (from 10 to 500 mils).

CHAPTER 2

TRAVELING WAVES

1. The Open-circuited Line. All computation of transmission-line operation must begin with conditions at the load. The relation of the reflected wave to the incident wave at every point of the line is determined by the nature of the load, and it is the receiving-end impedance (and not at all the sending-end impedance) that determines voltage and current distribution along a line.

Three special receiving-end conditions are particularly important, and will be considered individually. Then the general case, in which the load can have any impedance $\mathbf{Z}_r$, will be discussed.

One important termination is open circuit: the line has no load, and there can be no current at the receiving end. Calling the receiving-end current I_r, as in Eq. 1-31, it is true on any transmission line that

$$I_r = I_1 + I_2 \tag{2-1}$$

On an open-circuited line, for which $I_r = 0$,

$$I_1 + I_2 = 0 \tag{2-2}$$

from which

$$I_2 = -I_1 \tag{2-3}$$

In words, the current of the incident wave and the current of the reflected wave must be equal and opposite at the receiving end of an open-circuited line, for they must add to zero at the open-circuited point.

The corresponding voltage relation is found by substituting Eqs. 1-46 and 1-47 into Eq. 2-3, giving

$$V_2 = V_1 \tag{2-4}$$

That is, the voltage of the incident wave and the voltage of the reflected wave must be equal and in phase with each other at the receiving end of the line. The total receiving-end voltage will be twice the voltage of either traveling wave alone:

$$V_r = V_1 + V_2 = 2V_1 = 2V_2 \tag{2-5}$$

Figure 2-1*a* shows the incident and reflected voltages represented by identical vectors. Figure 2-1*b* is a vector diagram of voltages at a point on the line that is $\frac{1}{8}$ wavelength distant from the receiving end. The voltage produced at this point by the incident traveling wave is $\frac{1}{8}$ cycle ahead of the receiving-end voltage, for the incident wave reaches this point before it reaches the receiving end of the line. Correspondingly, the vector showing the voltage of the incident wave is advanced by an angle of $\frac{1}{8}$ cycle (or 45 degrees). The voltage of the reflected wave is retarded by the same angle. The total voltage at $\frac{1}{8}$ wavelength from the receiving end, marked $V_{\frac{1}{8}\lambda}$, is the vector sum of these two components. If the traveling waves were not attenuated as they travel, the voltage at this point would be in phase with the receiving-end voltage but smaller in magnitude: it would be, from the geometry of the figure, $V_{\frac{1}{8}\lambda} = 0.707V_r$. Actually, the traveling waves will be attenuated somewhat by energy loss along the line; if the attenuation is small the voltage at the $\frac{1}{8}$-wavelength point is very nearly the same as if there were no attenuation.

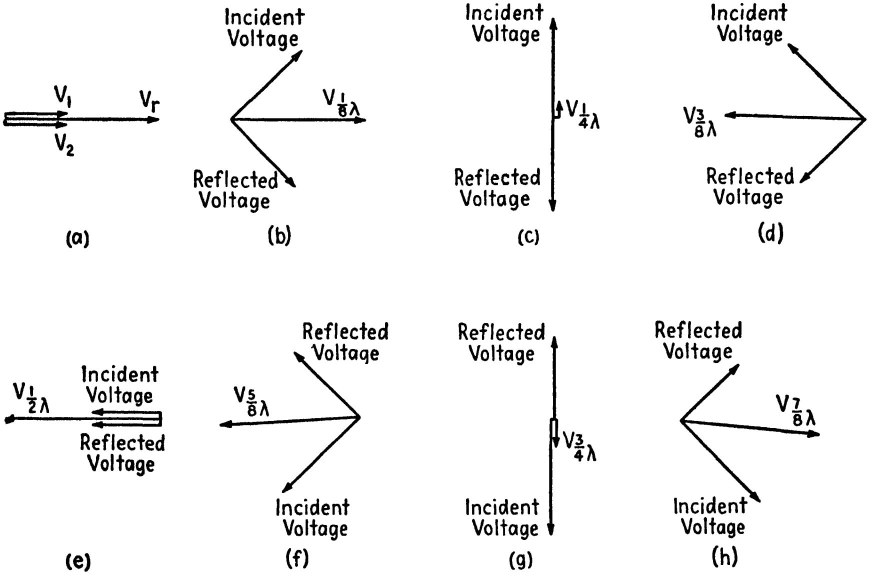

FIG. 2-1. Voltage of waves on an open-circuited line.

Figure 2-1*c* shows voltage vectors at a point $\frac{1}{4}$ wavelength from the receiving end. The incident-wave voltage at this point leads the incident-wave voltage at the receiving end by 90 degrees, and the reflected-wave voltage lags by an equal angle. The voltages of the two waves at the quarter-wavelength point are therefore opposite to each other in phase;

if they were equal in magnitude the total voltage at the quarter-wavelength point would be zero. Because attenuation makes the reflected wave slightly smaller than the incident wave, the sum of the two is the small voltage marked $V_{\frac{1}{4}\lambda}$ in the figure. It will sometimes be convenient to neglect attenuation on a transmission line, and if this is done the voltage ¼ wavelength from the end of an open-circuited line is found to be zero. Such a conclusion does not seem to be reasonable, and as a matter of fact it is easily seen that this is a limiting condition than can be approached, but never quite reached, as line loss is diminished toward zero.

Figure 2-1*d* shows voltage at a point ⅜ wavelength from the end of the line. Each component voltage differs in angle by 135 degrees from its phase at the receiving end. The magnitude of the resultant is nearly the same as at ⅛ wavelength, but the phase of the total voltage is reversed.

Consideration of vector diagrams drawn for a succession of points quite close to the quarter-wavelength point will show that the reversal of phase of the voltage takes place quite rapidly in this region. If there were no attentuation, the total voltage would be exactly in phase with the receiving-end voltage until the quarter-wavelength point was reached, and just at that point the phase would be suddenly reversed. On any actual line, most of the shift of phase may take place in a short distance near the quarter-wavelength point, but the reversal is never absolutely abrupt.

At ½ wavelength the two components of voltage are again in phase, as shown in Fig. 2-1*e*, and the total voltage, $V_{\frac{1}{2}\lambda}$, is opposite in phase but nearly equal in magnitude to the receiving-end voltage. Because of attenuation, the reflected voltage at the half-wavelength point is less than V_2, which is its value at the receiving end, and the incident voltage is greater than V_1. If the reduction in one component were equal to the increase of the other, total voltage at the half-wavelength point would just equal V_r; this would be true if attenuation were linear with distance. Actually, however, attenuation is an exponential function of distance, with the result that $V_{\frac{1}{2}\lambda}$ is slightly greater than V_r.

Figures 2-1*f* to *h* show voltage vectors at ⅝, ¾, and ⅞ wavelength, respectively. Such a series of vector diagrams could be continued indefinitely for a line of any length. Some lines are many wavelengths long; this is particularly true of lines operating at high radio frequency, for which the wavelength is short. Except that the effect of attenuation is cumulative, the diagrams of Fig. 2-1 will be similar to the diagrams that apply in the second wavelength of the line, and again in the third, and so on. It may be specifically mentioned that if there were a diagram to follow Fig. 2-1*h*, to show voltages at a point 1 wavelength from the receiving end, it would be very similar to Fig. 2-1*a*. Indeed, it is a valuable generalization that conditions on any line *with zero losses* are exactly repeated every wavelength.

Figure 2-2 shows current vectors at the same points on the same line. At the receiving end, I_1 and I_2 are equal and opposite as in Eq. 2-3, and total current is zero. The phase of the current vectors relative to the voltage vectors must be investigated. We have, from Eqs. 1-46 and 1-47,

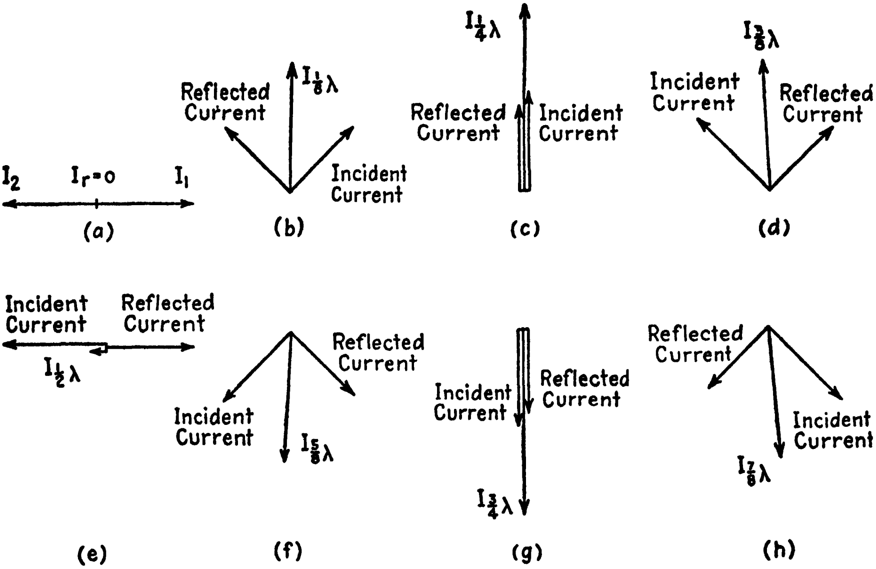

FIG. 2-2. Current of waves on an open-circuited line.

$$V_1 = Z_0 I_1 \tag{2-6}$$

$$V_2 = -Z_0 I_2 \tag{2-7}$$

On most ordinary transmission lines, Z_0, the characteristic impedance, is very nearly a real number. By definition, from Eq. 1-38,

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \tag{2-8}$$

If

$$r = 0 \quad \text{and} \quad g = 0 \tag{2-9}$$

Z_0 is real. Z_0 is approximately real on any line with low loss, for which r and g are small compared with ωl and ωc; on such a line the angle of Z_0 cannot be large.

Since Figs. 2-1 and 2-2 are illustrative of lines with small losses and consequent low attenuation, it will be appropriate to assume that I_1 is nearly in phase with V_1 and that I_2 is nearly opposite V_2. Figure 2-2*a* results.

Figure 2-2*b* shows current at $\frac{1}{8}$ wavelength. At this point, as at the receiving end (and, indeed, at every point on the line), the current components are related to their respective voltage components in both magnitude and angle by the characteristic impedance. Hence the incident current is in phase with the incident voltage (or nearly so) in Fig. 2-2*b*, and the reflected current is opposite to the reflected voltage. The total current at this point, $I_{\frac{1}{8}\lambda}$, is in quadrature with the total voltage at the same point—or, if loss is not entirely negligible, the current and voltage are *nearly* in quadrature. Considering power, this quadrature relation is necessary; there is no load to consume power, and little power is dissipated in the line. Power transmitted past any point of the open-circuited line must be just enough to supply losses in the rest of the line, between that point and the open end. Hence if voltage and current are both moderately large, as at the $\frac{1}{8}$-wavelength point, they must be nearly in quadrature; whereas if voltage and current are in phase, as at the quarter- and half-wavelength points, one or the other must be very small.

It is important to notice that current leads voltage at the point $\frac{1}{8}$ wavelength from the open end. The current is really charging current to the capacitance of the open-circuited section of line, and it is natural that such current, like the charging current to any condenser, leads the voltage.

Figure 2-2*c* shows current at $\frac{1}{4}$ wavelength. Here the incident and reflected components are in phase, making the total current maximum. This is the point at which voltage is minimum or, on an ideal line, zero.

Figure 2-2*d* shows current at $\frac{3}{8}$ wavelength. The magnitude of total current is less than the maximum value reached at $\frac{1}{4}$ wavelength, but the phase remains nearly the same. The phase of voltage at this point, however, has reversed from the phase of all voltages in the first quarter-wavelength of line. As a result, current at the $\frac{3}{8}$-wavelength point is lagging, instead of leading, the voltage at that point.

It was stated in a preceding paragraph that the current, being charging current, will naturally lead the voltage. This is true when the open-circuited line is short enough. On a longer line, however, the current must flow through so much inductance (the distributed inductance of the line) to reach the section of the line that receives it as charging current that the effect is predominantly inductive, and the current lags the voltage that drives it. The situation is analogous to charging a condenser through a coil if the inductive reactance of the coil is greater than the capacitive reactance of the condenser. From this point of view, it is not surprising that an intermediate length of line provides a kind of resonance, with inductive and capacitive effects balancing each other; this resonant condition is found at the quarter-wavelength point, at which the voltage could be zero (as in a series resonant circuit) if the losses were zero.

The Polar Diagram. It is interesting to collect the voltage and current vectors of Figs. 2-1 and 2-2 in a single diagram, as in Fig. 2-3. In this diagram the magnitudes and phase angles of the voltages and currents at different distances along the line appear in relation to each other. A curve is drawn, an elliptic spiral, through the tips of the voltage vectors; this is a locus of voltages at all points along the line. A similar curve, drawn as a dash line, is a locus of currents at all points. The discussion of voltage and current distribution on an open-circuited line may well be reviewed with Fig. 2-3 in mind.

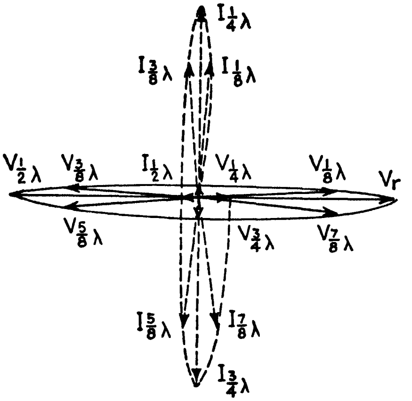

FIG. 2-3. Voltage and current loci on an open-circuited line.

2. The Short-circuited Line. Another receiving-end condition of exceptional importance is short circuit. When the receiving end of the line is short-circuited, voltage must be zero where $x = 0$. Making x and V both zero in Eq. 1-47 gives

$$V_1 + V_2 = 0 \tag{2-10}$$

or

$$V_2 = -V_1 \tag{2-11}$$

Substituting into this relation, Eqs. 2-6 and 2-7, gives

$$I_2 = I_1 \tag{2-12}$$

From these equations, conditions along the line are determined.

Vector diagrams of current and voltage at various points along the line can be drawn for a short-circuited line, similar to those that were drawn for the open-circuited line in Figs. 2-1 to 2-3. Indeed, the similarity is so great that Figs. 2-1 to 2-3 can actually be used as applying to the short-circuited line if we interchange current and voltage. That is, the diagrams of Fig. 2-1, which were drawn for voltage on an open-circuited line, represent equally well the current in a short-circuited line; and the diagrams of Fig. 2-2, intended for current on an open-circuited line, can be used for voltage on a short-circuited line. (This assumes that Z_0 is real, or nearly so.)

In general, the statements made in the discussion of voltage distribution on an open-circuited line may be applied to current distribution on a short-circuited line, and vice versa. The way in which magnitude and phase change, the repetitive nature of the voltage and current in successive wavelengths, and the necessity of transmitting power only to supply line losses are characteristic of the short-circuited line as they are of the open line.

It was mentioned that current at a point $\frac{1}{8}$ wavelength from the end of an open-circuited line leads the voltage at that point. The line being open, the current is charging current to the capacitance of the line, and the input impedance to an open-circuited line $\frac{1}{8}$ wavelength long is capacitive. Similarly, the current at a point $\frac{1}{8}$ wavelength from the end of a *short*-circuited line lags relative to the voltage at that point, and the input impedance is inductive.

The following comparison may help one remember. A short length of transmission line, if short-circuited, is a loop of a single turn; if open-circuited, it is a condenser of two parallel conductors.

The input impedance to a short-circuited line is inductive if the line is short; it is capacitive, however, if the length is greater than $\frac{1}{4}$ wavelength but less than $\frac{1}{2}$ wavelength.

Figure 2-3 shows the phase relation of voltage and current on an open-circuited line, and impedance (the ratio of voltage to current) is easily deduced. The same diagram may be used for a short-circuited line if the V's and I's are interchanged.

3. The Line Terminated in Z_0. Another termination of particular interest is that which permits the line to carry only an incident traveling wave of voltage and current, without requiring the existence of a reflected wave also.

The current of the incident traveling wave is related to the voltage of the incident wave by the characteristic impedance:

$$\frac{V_1}{I_1} = Z_0 \tag{2-13}$$

If there is no reflected wave, then the voltage and current of the incident wave are the total voltage and current, and the ratio of total voltage to total current at every point of the line is Z_0:

$$\frac{V}{I} = Z_0 \tag{2-14}$$

Since this is true at every point on the line, it must be true at the receiving end of the line, so it is necessary that the ratio of receiving-end voltage to receiving-end current be Z_0:

$$\frac{V_r}{I_r} = Z_0 \tag{2-15}$$

However, the ratio of V_r to I_r is the load impedance, for I_r is the current through the load and V_r is the voltage across the load:

$$\frac{V_r}{I_r} = Z_r \tag{2-16}$$

Hence, for the incident wave to exist alone on the line, the load impedance must equal the characteristic impedance:

$$Z_r = Z_0 \tag{2-17}$$

This suggests an interesting interpretation of transmission-line operation. If a line is correctly terminated, so that the receiving-end impedance matches the characteristic impedance of the line, the incident wave will be entirely absorbed by the load and there will be no reflection.

If, however, the receiving-end impedance has any value other than Z_0, the current and voltage in the load cannot match the current and voltage of the incident wave on the line, and there will be a reflected wave sent back along the line. The size of the reflected wave will be just great enough to make the ratio of total voltage to total current at the receiving end of the line equal the load impedance.

Considering the problem of reflection from a slightly different point of view, the incident wave brings energy from the generator to the load. The rate at which the load can absorb energy is determined by the load impedance. If the impedance of the load matches the characteristic impedance of the line, the load will absorb all the energy supplied by the incident wave and there will be no reflection. If the impedance of the load does not match the characteristic impedance of the line, the load is unable to absorb energy as fast as it arrives, and some of the energy is rejected in the form of a reflected wave that returns the rejected energy toward the generator.[1]

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[1] This concept is often useful, but it is exact only in special cases. The lossless line, treated in this chapter, is one of the special cases in which the concept is correct. The basic fallacy lies in applying the principle of superposition to a nonlinear quantity, power. It is not, in general, permissible to associate a certain power with the incident wave and a certain power with the reflected wave. Rather, the total received power is

$$\begin{aligned} P_r &= \tfrac{1}{2}(\bar{V}_r I_r + V_r \bar{I}_r) \\ &= \tfrac{1}{2}[(\bar{V}_1 + \bar{V}_2)(I_1 + I_2) + (V_1 + V_2)(\bar{I}_1 + \bar{I}_2)] \\ &= \tfrac{1}{2}(\bar{V}_1 I_1 + \bar{V}_2 I_2 + \bar{V}_1 I_2 + \bar{V}_2 I_1 + V_1 \bar{I}_1 + V_2 \bar{I}_2 \\ &\qquad + V_1 \bar{I}_2 + V_2 \bar{I}_1) \end{aligned} \tag{2-18}$$

wherein $\bar{V}$ and $\bar{I}$ are the conjugates of V and I. The power that would be associated

A line terminated in its characteristic impedance has a uniquely simple distribution of current and voltage. There is only an incident traveling wave, and if this travels along the line without attenuation, it provides the same magnitude of voltage at every point on the line. If there is loss as the wave travels, the traveling wave is attenuated, and voltage and current are somewhat smaller near the load than near the generator. If there is attenuation, voltage and current are both attenuated as the wave travels; they are attenuated at the same rate, and the ratio of voltage to current is the same at every point along the line, and is equal to Z_0.

Phase relations are also simple on a line terminated in its characteristic impedance. Since only the incident wave exists, the phase relations are those of the incident wave; that is, the phase angle of voltage at any point is proportional to the distance of that point from the load end of the line (as in Fig. 1-6). The phase angle of current at any point is likewise proportional to distance from the load; and the angle between voltage at any point and current at that same point is constant. Indeed, since voltage and current are related by the characteristic impedance of the line, the angle between the voltage vector at any point and the current vector at that same point is the angle of the characteristic impedance.

Since the ratio of voltage to current at every point of the line is equal to the characteristic impedance, it is evident that voltage and current will have this same ratio at the sending end of the line. Hence, if a line is terminated in impedance Z_0, the input impedance to that line is equal to Z_0, and this is entirely independent of the length of the line.

For many purposes there are great practical advantages in having a line loaded with an impedance equal to its characteristic impedance. The

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with the incident wave if it were alone, and the power associated with the reflected wave if it were alone, are, respectively,

$$P_1 = \tfrac{1}{2}(\overline{V}_1 I_1 + V_1 \overline{I}_1) \qquad (2\text{-}19)$$

$$P_2 = \tfrac{1}{2}(\overline{V}_2 I_2 + V_2 \overline{I}_2) \qquad (2\text{-}20)$$

Comparing terms, $P_1 + P_2 = P_r$ only if

$$\overline{V}_1 I_2 + \overline{V}_2 I_1 + V_1 \overline{I}_2 + V_2 \overline{I}_1 = 0 \qquad (2\text{-}21)$$

Introducing Z_0 to eliminate current, the condition becomes

$$-\frac{\overline{V}_1 V_2}{Z_0} + \frac{V_1 \overline{V}_2}{Z_0} - \frac{V_1 \overline{V}_2}{\overline{Z}_0} + \frac{\overline{V}_1 V_2}{\overline{Z}_0} = 0 \qquad (2\text{-}22)$$

This condition is met only if $\overline{Z}_0 = Z_0$, or if $\overline{V}_1 V_2 = V_1 \overline{V}_2$, which require that either Z_0 be real or V_2/V_1 be real. Hence total power is equal to the power of the two traveling waves considered independently if either Z_0 is a purely real quantity (see Eq. 2-9), or else V_2/V_1 is real, which requires (by Eq. 2-29) that Z_r/Z_0 be real.

termination of a power line must, of course, depend on the demand for power—the amount of connected load on the customers' premises. But communication lines can be, and nearly always are, terminated in their characteristic impedance. This provides for the greatest power transfer from line to load, as reflected waves which carry energy in the wrong direction are eliminated. It also provides that input impedance to the line, regardless of length, will be equal to the characteristic impedance. Other advantages of a line so terminated will be discussed in Chap. 9.

4. The Reflection Factor. If a line is terminated in any impedance except Z_0, there will be a reflected wave as well as an incident wave. The ratio of the voltage of the reflected wave to the voltage of the incident wave at the receiving end of the line is called the *reflection factor* or coefficient. To find the reflection factor, Eqs. 1-30 and 1-31 are repeated:

$$V_r = V_1 + V_2 \tag{2-23}$$

$$I_r = I_1 + I_2 \tag{2-24}$$

These are substituted into Eq. 2-16 to give

$$Z_r = \frac{V_r}{I_r} = \frac{V_1 + V_2}{I_1 + I_2} = \frac{V_1 + V_2}{\dfrac{V_1}{Z_0} - \dfrac{V_2}{Z_0}} = Z_0 \frac{V_1 + V_2}{V_1 - V_2} \tag{2-25}$$

Solution for V_2 leads to the reflection factor:

$$\frac{V_2}{V_1} = \frac{Z_r - Z_0}{Z_r + Z_0} \tag{2-26}$$

The reflection factor gives V_2 as a fraction of V_1. Sometimes it is more useful to be able to find both V_1 and V_2 in terms of the total receiving-end voltage V_r. This is easily done by substituting Eq. 2-26 into Eq. 2-23, first to eliminate V_2 and then to eliminate V_1. There result

$$V_1 = V_r \frac{Z_r + Z_0}{2Z_r} \tag{2-27}$$

$$V_2 = V_r \frac{Z_r - Z_0}{2Z_r} \tag{2-28}$$

These will be useful in a later section.

It is often helpful to think of load impedance, not as so-many ohms, but as some fraction or multiple of the characteristic impedance of the line. That is, the interesting quantity is not Z_r but is Z_r/Z_0. This ratio is in general complex. When this ratio is the matter of interest, a better form of Eq. 2-26 is

$$\text{Reflection factor} = \frac{V_2}{V_1} = \frac{\dfrac{Z_r}{Z_0} - 1}{\dfrac{Z_r}{Z_0} + 1} \tag{2-29}$$

If reflection factor is designated by ρ and Z_r/Z_0 by ζ (zeta) this equation becomes

$$\rho = \frac{\zeta - 1}{\zeta + 1} \tag{2-30}$$

and this form will be used later.

Examples of reflection at the ends of lines terminated in open circuit, short circuit, and in a load impedance equal to characteristic impedance were considered in the early part of this chapter. It will now be shown that these are merely special cases that may be deduced from Eqs. 2-26 and 2-29.

For the line terminated in its characteristic impedance. $Z_r/Z_0 = 1$, and by Eq. 2-29 the reflection factor is zero; hence there is no reflected wave.

For the line that is short-circuited at the receiving end, $Z_r = 0$, and Eq. 2-29 gives the reflection factor as -1; hence the reflected voltage is equal to the incident voltage, but of opposite sign.

For the line that is open-circuited, $Z_r = \infty$. The reflection factor is $+1$. (This is most readily deduced from Eq. 2-26 after dividing numerator and denominator of the right-hand member by Z_r.) The reflected voltage is therefore equal to the incident voltage.

By way of further illustration, a few more special values of reflection factor may be computed. If $Z_r/Z_0 = 3$ (an illustration of load impedance greater than characteristic impedance), the reflection factor is $\frac{1}{2}$.

If $Z_r/Z_0 = \frac{1}{3}$ (an illustration of load impedance less than characteristic impedance), the reflection factor is $-\frac{1}{2}$.

If $Z_r = jZ_0$, the reflection factor is j. This means that if we may assume Z_0 to be purely real, and if the load impedance has the same value in ohms as the characteristic impedance of the line, but is purely inductive reactance, the reflected wave has the same magnitude as the incident wave, but at the receiving end of the line the voltages of the two waves are out of phase by 90 degrees and $V_2 = jV_1$.

Similarly, if $Z_r = -jZ_0$, a load with capacitive reactance, the reflection factor is $-j$.

It is seen that the reflection factor may be imaginary or complex. In general, Eq. 2-29 shows that the reflection factor is real on a line with real Z_0 if the load is purely resistive. Since the resistive load must, from the nature of resistance, be a positive quantity, Eq. 2-29 also shows that the reflection factor is never numerically greater than 1. This means that the reflected wave cannot be greater than the incident wave, which is reasonable if the reflected wave contains only that part of the incident energy that is rejected by the load.[2]

Line with Resistance Load. Let us return to give further consideration to a line with a receiving-end impedance $Z_r = 3Z_0$. As was said, the reflection factor is $\frac{1}{2}$. With a reflection factor of $\frac{1}{2}$, the reflected voltage is half the incident voltage. Vector diagrams in Fig. 2-4 show voltage and current at the receiving end of the line and at points along the line at which $x = \lambda/12$, $x = \lambda/6$, and $x = \lambda/4$. It is convenient for some purposes, particularly in radio-frequency work, to identify these points by the value of βx expressed in degrees, and refer to them as points at 30, 60, and 90 degrees, respectively; it is understood that 90 degrees is $\frac{1}{4}$ wavelength, and 360 degrees is 1 wavelength.

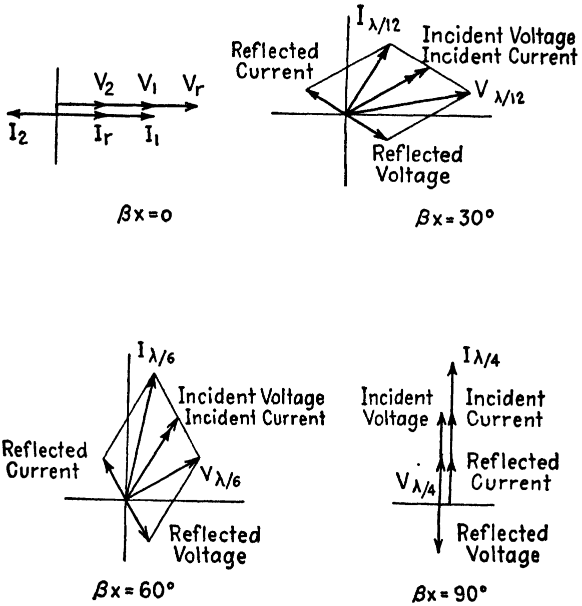

FIG. 2-4. Vector diagrams, $Z_r = 3Z_0$.

The diagrams of Fig. 2-4 are somewhat similar to the diagrams of

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[2] More generally, the reflection factor is real if Z_r/Z_0 is real. The reflection factor cannot be greater than 1 if the real component of Z_r/Z_0 is positive, and this is true in nearly all practical cases although special conditions can be devised in which it is not true and which actually provide a reflected wave of greater magnitude than the incident wave. The reflected wave may be greater in voltage and greater in current but not, of course, greater in power; component power is meaningless in such a case.

Figs. 2-1 and 2-2, but there are significant differences. At the receiving end of the line, the current, as in Fig. 2-4, is small, but it is not zero. At ¼ wavelength the current is large, while the voltage is small, but the voltage is by no means as small as in Fig. 2-1*c*. If power is determined in the diagrams of Fig. 2-4, it will be found to be nearly the same in each, for at each point the power is equal to that delivered to the load, plus the power loss (assumed to be relatively small) in the part of the line that extends from the point under consideration to the load.

Total voltage and current vectors from Fig. 2-4 are collected in Fig. 2-5, and vectors for another quarter-wavelength of line are also included in this figure. It is instructive to compare this diagram for a line with a resistive load to Fig. 2-3 which relates to an open-circuited line.

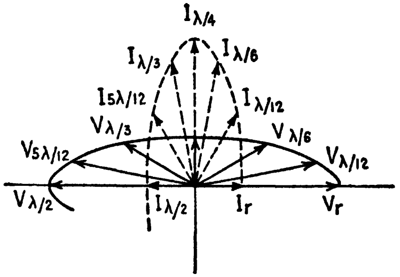


FIG. 2-5. Voltage and current loci, $Z_r = 3Z_0$.

Line with Reactive Load. If the load on a transmission line is purely reactive, either inductive or capacitive, it will not consume any energy. As a result, the reflected wave on the line (assuming Z_0 is real) is as large as the incident wave, for all the energy of the incident wave is rejected and must be carried away by the reflected wave.

The reactive load, however, although it cannot consume energy, is able to store energy for part of a cycle and then return it to the line as a reflected wave that is not in phase with the incident wave. This is consistent with the determination that the reflection factor of an inductive reactive load for which $Z_r = jZ_0$ is j, and the reflection factor of a capacitive reactive load for which $Z_r = -jZ_0$ is $-j$.

The vector diagram of Fig. 2-6*a* shows the relation that must be true in order to satisfy both line conditions and load conditions when $Z_r = -jZ_0$, a capacitive load. As indicated in the figure, I_r leads V_r by 90 degrees, V_1 and I_1 are in phase (for Z_0 is assumed real), and V_2 and I_2 are opposite. The reflection factor is seen to be $-j$ because V_2 is lagging V_1 by 90 degrees.

Figure 2-6*b* shows vectors of voltage and current at a distance of ⅛ wavelength from the load end of the line. Since the incident and reflected components of current are in phase at this point, this is a point of maximum current in the line. Since incident and reflected voltage are opposite, and would be equal except for attenuation, it is a point of minimum voltage. Voltages and currents at other points on the line can be found by drawing a series of such diagrams.

It was stated above that the reflected wave is as large as the incident wave if the load is purely reactive and Z_0 is real. A little more generally, if Z_r/Z_0 is imaginary the magnitude of the reflection factor is 1. To see that this is true, consider Fig. 2-7. Z_r/Z_0 is plotted as a point in the complex plane, and if it is imaginary it will fall somewhere on the vertical axis. The denominator of Eq. 2-29 is obtained by adding 1 to this value, and the numerator by subtracting 1. The reflection factor is the ratio of these two complex quantities, which are shown as vectors, and since the vectors are of the same length, the magnitude of the ratio is 1. The angle of the reflection factor is the angle between the two vectors, 2θ, where θ is the angle shown whose tangent is $|Z_0|/|Z_r|$.

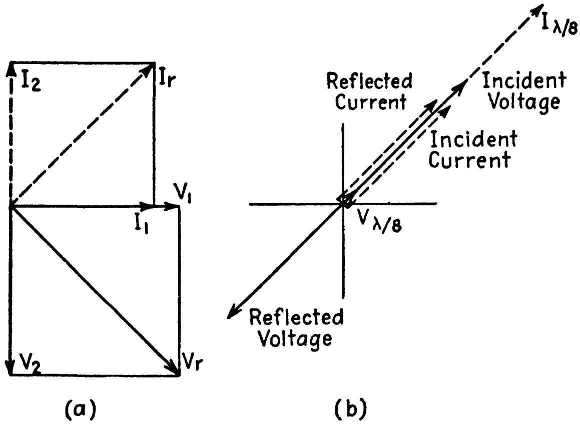


FIG. 2-6. Vector diagrams, $Z_r = -jZ_0$.

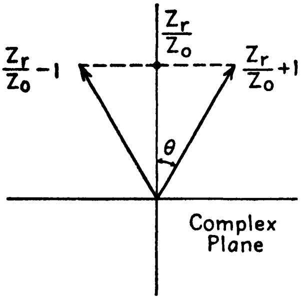


FIG. 2-7. If ζ is imaginary, $|\rho| = 1$.

Hence the wave reflected at a purely reactive load is equal in amplitude to the incident wave, but the reflected wave is out of phase with the incident wave by an angle dependent upon the load reactance.

Line with Load of Complex Impedance. Although only purely real or purely imaginary load impedances have yet been considered, the load impedance can, in general, have a complex value.

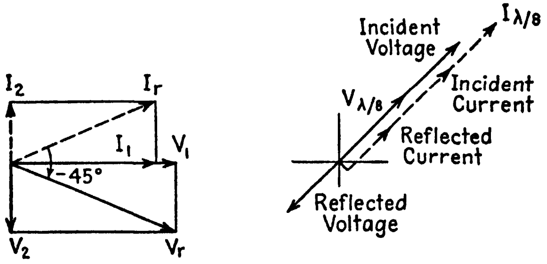
It would not be fruitful to try to discuss the entire range of complex impedances, but an interesting cross section is obtained by studying the behavior of the line when the magnitude of the load impedance is required to be equal to the magnitude of Z_0 while the angle of the load impedance is permitted to vary. That is, if φ is the difference between the angle of the load impedance and the angle of the characteristic impedance,

$$\frac{Z_r}{Z_0} = e^{j\varphi} \tag{2-31}$$

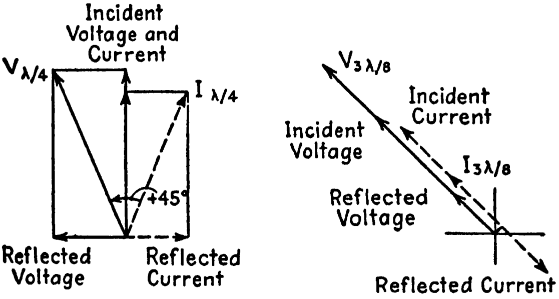
With this restriction, the reflection factor, from Eq. 2-29, is

$$\frac{V_2}{V_1} = \frac{e^{j\varphi} - 1}{e^{j\varphi} + 1} = j\frac{\sin\varphi}{\cos\varphi + 1} \tag{2-32}$$

This result has two interesting implications. First, the reflection factor is always imaginary, meaning that V_2 is in quadrature with V_1 if $|Z_r| = |Z_0|$. From this it follows that a line so terminated will always have a point of either maximum or minimum voltage $\frac{1}{8}$ wavelength from the load; consider Fig. 2-6.



(a) Receiving End (b) Voltage Minimum



(c) Quarter Wavelength (d) Voltage Maximum

FIG. 2-8. Vector diagrams, $Z_r = Z_0 \underline{/-45°}$.

Second, this type of termination is identical in three special cases with terminations that have been discussed earlier. If $\varphi = \pi/2$ the load is an inductive reactance; the reflection factor is j. If $\varphi = -\pi/2$ the load is a capacitive reactance, as illustrated in Fig. 2-6; the reflection factor is $-j$. If $\varphi = 0$, the load impedance is equal to the characteristic impedance of the line, the reflection factor (see Eq. 2-32) is zero, and there is no reflected wave.

As a single example, let us consider a line with a load such that $Z_r = Z_0\underline{/-45^\circ}$. Vector diagrams for conditions at the receiving end and at $\frac{1}{8}$, $\frac{1}{4}$ and $\frac{3}{8}$ wavelength are shown in Fig. 2-8. Note the voltage minimum at $\frac{1}{8}$ wavelength, and the corresponding maximum at $\frac{3}{8}$ wavelength. If a diagram similar to Fig. 2-5 is drawn for a line with this termination, it gives an interesting comparison. Note also the imaginary reflection factor, indicated in Fig. 2-8*a* by the 90-degree angle between V_1 and V_2, and the way in which the voltage and current components combine at the receiving end to produce the necessary relations between load voltage and load current.

5. Line Impedance. The input impedance to a transmission line is the ratio of the voltage at the sending end of the line to the current entering the line at the sending end. It is obvious that if a line is exceedingly short, the input impedance will hardly differ from the load impedance. It has been mentioned that if a line is terminated in its characteristic impedance, the input impedance to a line of any length will also be equal to the characteristic impedance, but this is a special case, and in general the input impedance will be different from the load impedance.

The input impedance to a line can, of course, be measured experimentally. An impedance bridge, or suitable instruments for measuring current, voltage, and phase angle, will give the necessary data. The input impedance can be computed by dividing voltage by current.

By an extension of the idea of impedance at the sending end of the line, it is customary to speak of impedance at any point of a transmission line. The impedance at any point is defined as the ratio of voltage to current at that point. Such an impedance can be found experimentally by measuring voltage, current, and relative angle at the point in question, or it can be measured on a bridge by cutting the line at the desired point and measuring the input impedance to the part of the line that leads to the load.

For purposes of computation, the impedance is to be found from

$$Z = \frac{V}{I} \tag{2-33}$$

V, the voltage at a point on the line, is the sum of the voltage of the incident wave and the voltage of the reflected wave at that point, and I, the current, is the sum of the current components; using Eqs. 1-41 and 1-43 in 2-33,

$$Z = \frac{V}{I} = \frac{V_1 e^{\gamma x} + V_2 e^{-\gamma x}}{I_1 e^{\gamma x} + I_2 e^{-\gamma x}} = \frac{V_1}{I_1} \frac{1 + \frac{V_2}{V_1} e^{-2\gamma x}}{1 + \frac{I_2}{I_1} e^{-2\gamma x}} \tag{2-34}$$

V_1/I_1 is Z_0, the characteristic impedance. V_2/V_1 is, by definition, the reflection factor ρ. From Eqs. 2-6 and 2-7 it is evident that I_2/I_1 is $-\rho$. Hence

$$\frac{Z}{Z_0} = \frac{1 + \rho e^{-2\gamma x}}{1 - \rho e^{-2\gamma x}} \tag{2-35}$$

where, by Eq. 2-30,

$$\rho = \frac{\frac{Z_r}{Z_0} - 1}{\frac{Z_r}{Z_0} + 1} \tag{2-36}$$

This pair of equations gives a means of computing impedance at any point. First find ρ, the reflection factor at the load, by using the load impedance in Eq. 2-36. Then find Z, for any value of x, from Eq. 2-35.

This computation for impedance can be a little tiresome, although it is quite straightforward. The labor of computation is saved if a graphic solution is employed and it will be shown in Chap. 12 that charts can be evolved from Fig. 2-9, using Eqs. 2-35 and 2-36, to give Z/Z_0 very easily if the impedance at the load, Z_r/Z_0, is known. Certain charts for this purpose will now be mentioned.

6. The Complex Impedance Plane. In Fig. 2-7 the impedance ratio Z_r/Z_0 is plotted as a point in the complex plane. Since the impedance ratio in that case is purely imaginary, it is represented by a point on the vertical axis of imaginaries.

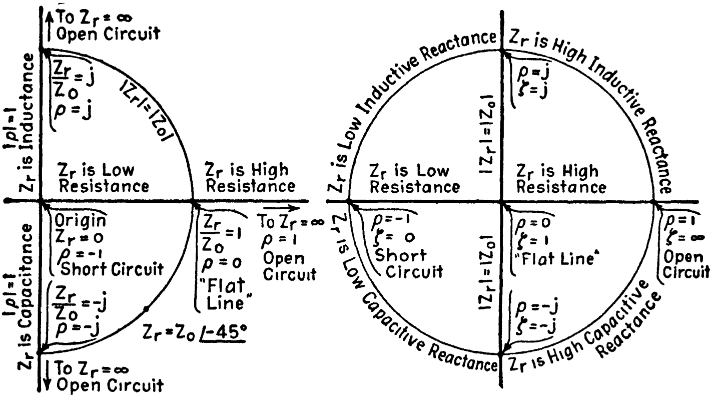
Similarly, other values of Z_r/Z_0, corresponding to all other load impedances, can be represented by other points in the complex plane. As a matter of fact, since Z_0 is usually real or approximately so, and since the resistance component of Z_r, the load impedance, can ordinarily have only a positive value, we can expect that the complex quantity Z_r/Z_0 will have a positive real component, and hence will be represented by a point somewhere in the right-hand half of the complex plane.

Figure 2-9*a* shows the location in the right-hand half of the complex plane of the line terminations that have been discussed in this chapter. All loads of pure resistance (or, more precisely, all loads for which Z_r/Z_0 is real) are represented by points on the horizontal axis. If the resistance of the resistive load is less than the characteristic impedance, it is represented by a point on the real axis between the origin and the point $Z_r/Z_0 =$

1; if the resistance is greater than Z_0, the corresponding point falls on the axis to the right of that point.

For every value of the ratio Z_r/Z_0, which may be called ζ, there is a corresponding value of the reflection factor ρ (as from Eq. 2-30). Values of the reflection factor are noted on the chart at the points at which Z_r/Z_0 equals zero, one, and infinity.

Points representing purely reactive loads (more precisely, loads for which Z_r/Z_0 is imaginary) fall on the vertical axis. If the reactance is low, the point is near the origin; if the load is capacitive, the point is below the origin; if inductive, above. For all points on this vertical axis, the reflection factor ρ has a magnitude of 1 and an angle that varies with the load reactance.



(a) Complex ζ Plane $\zeta = \frac{Z_r}{Z_0}$

(b) Complex ρ Plane $\rho = \frac{V_2}{V_1}$

FIG. 2-9. The impedance plane and the reflection-factor plane.

When the load impedance is equal in magnitude to Z_0, though it may differ in angle, the ratio ζ will have a magnitude of 1. Corresponding points in the ζ plane of Fig. 2-9*a* will lie on the semicircle of radius 1. The position on the semicircle will be determined by the angle of the load impedance.

The practical value of this chart in the ζ plane will be seen later. Another chart, even more useful in later work, can be made by plotting values of ρ, the reflection factor, in a complex plane. This is called the ρ plane to distinguish it from the ζ plane of Fig. 2-9*a*, and it is shown in Fig. 2-9*b*. The origin of this chart is the point $\rho = 0$, and this corresponds to the point on the ζ plane for which $\zeta = 1$. Real reflection factors, corresponding to resistive loads, lie to the right and left of the origin on the real axis, with short circuit at -1 and open circuit at $+1$. No useful points lie to the right of +1, or to the left of −1. Indeed, since all practical use of a chart of this kind is with lines that cannot have a reflection factor greater than 1 (see footnote 2), all the points of interest of the chart lie on or within the circle of radius 1 about the origin.

The point $\rho = 1$ is open circuit, and this is true whether the load is considered to be an infinite resistance or an infinite reactance. It has already been mentioned that all resistance loads correspond to points on the horizontal axis and that the point 1 is approached as the resistance becomes large without limit. If the load is purely reactive, all corresponding points lie on the circle shown in Fig. 2-9*b* which has radius 1 with center at the origin. Inductive loads correspond to points on the upper semicircle; capacitive loads to points on the lower semicircle. If the reactance of an inductive load is increased without limit, the point $\rho = 1$ is approached along the upper semicircle; if a capacitive reactance is increased without limit, the same point is approached along the lower semicircle.

As a crude but sometimes helpful concept, imagine the ζ plane of Fig. 2-9*a* to be a sheet of rubber. The left-hand part is now stretched, and the right-hand half is shrunk, so that the semicircle is bent into a straight line and the vertical axis is bent and distorted until it becomes a circle; the right-hand section of the real axis is shrunk so much that the point formerly at infinity becomes a point at +1, and by the same shrinking and bending operation the infinitely distant ends of the vertical axis are also brought to the point +1. The result is the diagram of Fig. 2-9*b*, the ρ plane. Everything in the right-hand half of the ζ plane is within the unit circle of the ρ plane. As will be discussed later, this is a type of conformal transformation, and the diagram in the ρ plane turns out to be useful in a form commonly known as the *Smith chart.*

PROBLEMS

2-1. A line has small loss of energy in conductor resistance, and negligible conductance from one conductor to the other. The line is open-circuited at the receiving end. Show that the power passing a point $\frac{3}{4}$ wavelength from the open-circuited receiving end is approximately three times the power passing a point $\frac{1}{4}$ wavelength from the end.

2-2. Several values of the ratio Z_r/Z_0 (the normalized receiving-end impedance) are given below. Find the corresponding values of the reflection factor. (*a*) 2; (*b*) $\frac{1}{2}$; (*c*) 4; (*d*) $\frac{1}{4}$; (*e*) $3\underline{/45^\circ}$; (*f*) $j/2$; (*g*) $1 - j1$; (*h*) $1\underline{/-30^\circ}$.

2-3. Z_0 of a radio-frequency line is 50 ohms (purely real). Find the values of the reflection factor that correspond to the following values of Z_r: (*a*) 100; (*b*) 25; (*c*) 125; (*d*) 20; (*e*) $100\underline{/-45^\circ}$; (*f*) $j75$; (*g*) $30 - j40$; (*h*) $50 + j50$.

2-4. Draw vector diagrams of current and voltage for a low-loss line on which

$Z_r = -jZ_0$, as in Fig. 2-6. Draw such diagrams for points at the following distances from the load: 0, $\frac{1}{8}\lambda$, $\frac{1}{4}\lambda$, $\frac{3}{8}\lambda$, $\frac{1}{2}\lambda$, and $\frac{5}{8}\lambda$.

2-5. Draw vector diagrams of current and voltage for a low-loss line on which $Z_r = jZ_0$ for points at the following distances from the load: 0, $\frac{1}{8}\lambda$, $\frac{1}{4}\lambda$, $\frac{3}{8}\lambda$, $\frac{1}{2}\lambda$, and $\frac{5}{8}\lambda$.

2-6. Derive Eq. 2-32.

2-7. Draw a diagram similar to Fig. 2-5 but for a line on which $Z_r = Z_0\underline{/-45^\circ}$. What is the reflection factor?

2-8. Draw a diagram similar to Fig. 2-5 for a line on which $Z_r = Z_0\underline{/45^\circ}$. What is the reflection factor?

2-9. Assume a transmission line to be without loss. The load is such that $Z_r = \frac{1}{2}Z_0$. Find the reflection factor. Using Eq. 2-35, find Z/Z_0, the normalized impedance, at a distance of $\frac{5}{16}$ wavelength from the receiving end.

2-10. On the line of Prob. 2-9, and with the same load, after finding the reflection factor, compute Z/Z_0, the normalized impedance, at a distance of 0.15 from the receiving end.

2-11. On a cross-section sheet representing the complex plane, as in Fig. 2-9a, locate the values of Z_r/Z_0 given in Prob. 2-2 or 2-3. On another sheet, as in Fig. 2-9*b*, locate the computed values of the corresponding reflection factors. Note which points lie on the axes or unit circles of the diagrams.

CHAPTER 3

STANDING WAVES ON LOW-LOSS LINES

1. Low-loss Lines. The application of transmission-line mathematics to actual lines is greatly simplified if the energy loss in the line is small. Approximate forms of the equations are then adequate for many purposes.

The parameters that represent loss in the line are r and g. There are three types of problems to be distinguished:

1. r and g can be considered zero.
2. r and g are to be taken into account, but they are so small that an approximate method is satisfactory.
3. r and g are so significant that approximate methods are not adequate.

For many purposes, r and g are negligible if they are very much smaller than ωl and ωc, respectively. This condition is satisfied if r and g are exceedingly small, and it is also satisfied if ωl and ωc are very large. Since ω is $2\pi f$, this criterion will ordinarily be satisfied if the frequency is high. For many radio-frequency problems, therefore, r and g are negligible.

If r and g can be considered zero, z and y from Eqs. 1-6 and 1-16 become

$$z = r + j\omega l \approx j\omega l \tag{3-1}$$

$$y = g + j\omega c \approx j\omega c \tag{3-2}$$

The characteristic impedance of the line is then merely

$$Z_0 = \sqrt{\frac{z}{y}} \approx \sqrt{\frac{j\omega l}{j\omega c}} = \sqrt{\frac{l}{c}} \tag{3-3}$$

and the propagation constant on a lossless line is

$$\gamma = \sqrt{zy} \approx \sqrt{(j\omega l)(j\omega c)} = j\omega\sqrt{lc} \tag{3-4}$$

The characteristic impedance is purely real, for l and c are always positive quantities. Similarly, the propagation constant is purely imaginary.

Since α and $j\beta$ are by definition the real and imaginary parts of the propagation constant, it follows that when r and g are neglected, and γ is purely imaginary,

$$\alpha = 0 \qquad \text{and} \qquad \beta = \omega\sqrt{lc} \tag{3-5}$$

The traveling-wave equations for a lossless line, α being zero, are

$$V = V_1 e^{j\beta x} + V_2 e^{-j\beta x} \tag{3-6}$$

$$I = I_1 e^{j\beta x} + I_2 e^{-j\beta x} \tag{3-7}$$

On such a line, waves travel without attenuation. The incident wave is represented by voltage vectors that are all the same length, but that are more retarded in phase at points nearer the load. Vectors representing the reflected wave, on the contrary, are more retarded in phase at points more distant from the load.

Velocity of wave propagation on a lossless line is simply

$$\text{Velocity} = \frac{\omega}{\beta} = \frac{1}{\sqrt{lc}} \tag{3-8}$$

and the wavelength is

$$\lambda = \frac{\text{velocity}}{f} = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{lc}} \tag{3-9}$$

A numerical value for the velocity of propagation on a lossless line is easily computed, for Eq. 3-8 shows that velocity is determined by the inductance and capacitance of the line, and these are found from the formulas of Table I.

As a specially simple illustration, let us find velocity when the frequency is high enough to make the skin-effect constant K_φ of Formula 1 in Table I practically zero. Using values of inductance and capacitance from formulas 1 and 4, and substituting into Eq. 3-8,

$$\text{Velocity} = \frac{1}{\sqrt{[1.482 \log (b/a) 10^{-3}][19.44/\log (b/a) 10^{-9}]}}$$

$$= \frac{10^6}{\sqrt{(1.482)(19.44)}} = 186{,}300 \text{ miles per second} \tag{3-10}$$

The formulas used for inductance and capacitance in this computation are in terms of distance in miles, so the velocity is in miles per second. If Formulas 3 and 6 or 7 and 8 had been used, the velocity would have been found to be 3.00 × 10^8, in units of meters per second.

This velocity of propagation will be recognized as being also the velocity of light. It is a limit approached by waves on transmission lines; it is an ideal value, for if the value of K_φ were considered, or if there were any dielectric or magnetic material present, or if line losses were considered, the velocity of propagation would be somewhat less than the velocity of light.

The wavelength on an ideal line is

$$\lambda = \frac{3 \times 10^8}{f} \quad \text{meters} \tag{3-11}$$

2. Standing Waves. We have seen that the mathematical solution of the basic transmission-line equations can be interpreted as representing the sum of two traveling waves. One, the incident wave, travels from the generator toward the load, and the other, called the reflected wave, travels from the load toward the generator. On a line without loss, these waves travel without attenuation, and are described by Eqs. 3-6 and 3-7.

A change in the form of the mathematics gives a new point of view, and leads to another interpretation of the transmission-line phenomena. Transformation of the exponential functions into trigonometric functions suggests that the current and voltage distribution on the transmission line can be considered in terms of standing waves.

To make this change, Eq. 3-6 is modified by substituting into it Eqs. 2-27 for V_1 and 2-28 for V_2. The result is

$$V = V_r \frac{Z_r + Z_0}{2Z_r} e^{j\beta x} + V_r \frac{Z_r - Z_0}{2Z_r} e^{-j\beta x} \tag{3-12}$$

Regrouping terms,

$$V = V_r\left(\frac{e^{j\beta x} + e^{-j\beta x}}{2} + \frac{Z_0}{Z_r}\frac{e^{j\beta x} - e^{-j\beta x}}{2}\right) \tag{3-13}$$

The first of these fractions is identical with cos βx. (Various trigonometric formulas are collected in Table II, inside back cover, for convenient reference.) The second fraction in Eq. 3-13 would be equivalent to sin βx if it had j in the denominator. Hence we multiply both numerator and denominator by j and obtain

$$V = V_r\left(\cos \beta x + j\frac{Z_0}{Z_r}\sin \beta x\right) \tag{3-14}$$

or, since $V_r/Z_r = I_r$,

$$V = V_r \cos \beta x + jI_rZ_0 \sin \beta x \tag{3-15}$$

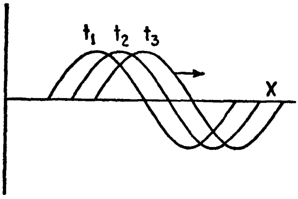
These are the trigonometric equations for voltage distribution. Current distribution is derived in the same way from Eq. 3-7, giving

$$I = I_r\left(\cos \beta x + j\frac{Z_r}{Z_0}\sin \beta x\right) \tag{3-16}$$

or

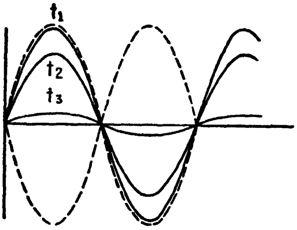
$$I = I_r \cos \beta x + j\frac{V_r}{Z_0}\sin \beta x \tag{3-17}$$

Equations 3-14 to 3-17 represent standing waves. The distinction between a traveling wave and a standing wave is shown in Fig. 3-1; Fig. 3-1*a* shows a traveling wave at three instants of time; the wave keeps the same size and shape as it moves along the line. Figure 3-1*b* shows a standing wave at three instants; it stays in one place and surges, up and down, each cycle. A standing wave has fixed *nodes*, or points of minimum amplitude, and *loops* of maximum amplitude.



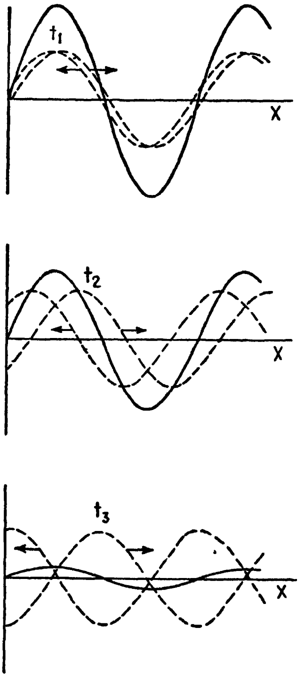
(a) One Traveling Wave

$v = \mathrm{Re}\left\{V_{max}\, e^{j(\omega t - \beta x)}\right\}$



(b) Standing Wave

$v = \mathrm{Re}\left\{2V_{max} \cos\beta x\, e^{j\omega t}\right\}$



(c) Two Traveling Waves

$v = \mathrm{Re}\left\{V_{max}\left(e^{j(\omega t - \beta x)} + e^{j(\omega t + \beta x)}\right)\right\}$

FIG. 3-1. Traveling waves combine to form a standing wave.

As in Fig. 3-1*c*, two traveling waves of the same amplitude, moving in opposite directions with the same speed, combine to give a standing wave. (If the two components were not equal in amplitude, the sum would not be a pure standing wave, but would be a combination of a traveling and a standing wave.)

The mathematical description of each type of wave is given in Fig. 3-1. The mathematical equivalence of *b* to *c* is apparent. For transmission-line work, as for other theory, it is customary to omit the symbol Re ("real part of") and, by dividing out the time variation (see footnote 2, Chap. 1), represent voltage (or current) by a complex quantity as in Eqs. 3-14 to 3-17, and in most of the other equations of this book. But in Fig. 3-1 the expressions for instantaneous voltage are fully written out to avoid any possible misunderstanding.

An envelope that contains the standing wave is shown in Fig. 3-1*b*. Hereafter when standing waves are to be represented, as in Figs. 3-2*a*, 3-3*a*, 3-6*a*, and so on, only this envelope will be shown. Using the envelope to represent the wave is equivalent to representing the wave by plotting the amplitude of the voltage vector at each point on the line.

3. Examples: Open Circuit. When the receiving end of the line is open-circuited, and $I_r = 0$, Eq. 3-15 becomes

$$V = V_r \cos \beta x \tag{3-18}$$

The value of β, a real number, can be computed from Eq. 3-5. V_r, we assume, is known. V, the voltage at any point, is a cosine function of the distance of the point from the receiving end of the line. Such a voltage distribution along the line is shown graphically in Fig. 3-2*a*.

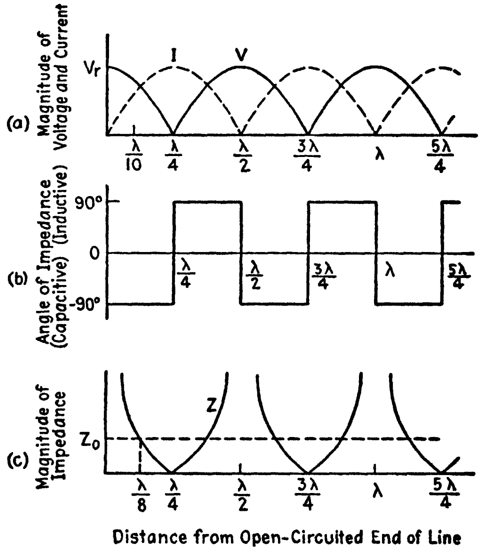


FIG. 3-2. Voltage, current, and impedance on open-circuited line.

The curve for voltage as plotted in Fig. 3-2*a* is the magnitude of the cosine function. The usual cosine curve indicates algebraic sign by plotting negative values below the axis, but in Fig. 3-2 algebraic sign is considered separately from amplitude.

The curve of voltage plotted in Fig. 3-2*a* can be considered a plot of the readings of an ordinary voltmeter at various points along the line. The curve is *not* a picture of a traveling wave. It shows the amplitude of a standing wave on the line.

Voltage nodes are seen at $\frac{1}{4}$ wavelength, $\frac{3}{4}$ wavelength, and the other odd quarter-wavelength points. At these nodal points there is no voltage. It is not at a particular instant only that the voltage is zero, but at all instants. A voltmeter connected from wire to wire of the line at a nodal point would read zero. This, it will be recalled, is assuming an ideally lossless line.

Current distribution along the open line is found from Eq. 3-17 to be

$$I = j\frac{V_r}{Z_0}\sin\beta x \tag{3-19}$$

The amplitude of current is shown as a dash line in Fig. 3-2*a*, and the curve has the familiar form of a sine curve. Since only amplitude is shown, however, the negative values are folded up and appear above the axis. Equation 3-19 indicates that the maximum amount of current is determined from the voltage and the characteristic impedance of the line. The current (because of the j) is everywhere in quadrature with the voltage, assuming, as we do, that the open-circuited line is ideally without loss.

In drawing or using these curves it is not necessary to know the actual length of the line. If the actual line is shorter than the distance for which the curves have been plotted, the excess length of the curves is merely disregarded. The plot may be cut off at any length, and the section remaining is entirely correct. Thus if the line is 1 wavelength long, everything to the right of the point λ is thrown away, and the voltage and current values at the point λ are the voltage and current at the sending end of the line. Note that in this special case, for which $x = \lambda$, the sending-end voltage and current are equal to voltage and current at the receiving end.

The longest existing power transmission lines are about one-tenth wavelength. For such lines, the curves of Fig. 3-2 are correct (neglecting line loss), but only the parts of the curves from the receiving end to the point $\lambda/10$ are significant—the rest of each curve is meaningless. At the generator end of the power line, which is at $\lambda/10$, the voltage is somewhat less than at the open-circuited receiving end. This rise of voltage at the receiving end of an open-circuited power line is sometimes called the Ferranti effect; it is obviously due to resonance between the capacitance and inductance of the line. The amount of charging current that the generator must supply to the open-circuited line may be read from the current curve at $\lambda/10$.

It will be seen that the voltage curve of Fig. 3-2*a* gives the voltage at any point in terms of the voltage at the receiving end. If the voltage at the receiving end is not known, and it often is not, the vertical scale of the graph has not been established, although the shape of the curve is known. Further information is needed, from which the scale can be found. If the actual voltage were known at any one point, that would be enough to give the voltage scale. Frequently the actual value of the sending-end voltage is known, and since the curve includes a point for sending-end voltage, this value is sufficient to determine the scale of the curve in volts.

The scale of the current curve is easily found when the voltage scale is known. From Eq. 3-19, the maximum current, at $\lambda/4$, is equal to the maximum voltage (at the receiving end) divided by the characteristic impedance of the line. Other values of current are then in proportion.

The ratio of voltage to current along the open line is found by dividing Eq. 3-18 by 3-19:

$$Z = \frac{V}{I} = -jZ_0 \cot \beta x \tag{3-20}$$

This will be recognized as the quantity that is called impedance, as in Sec. 6, Chap. 2. The magnitude of the impedance is plotted as Z in Fig. 3-2*c*; Eq. 3-20 indicates that the curve of Z is a cotangent curve. As Fig. 3-2*b*, there is plotted the angle of Z. If x is less than $\frac{1}{4}$ wavelength, βx is less than $\pi/2$, the cotangent function is positive, and Z is imaginary and negative. The angle of Z is therefore -90 degrees. If x is more than $\frac{1}{4}$ wavelength, but less than $\frac{1}{2}$ wavelength, the cotangent function is negative, making Z imaginary and positive. The angle of Z is then $+90$ degrees, as shown. It is evident that the angle of Z will continue to alternate in successive quarter-wavelengths. Thus the input impedance to an open-circuited line slightly less than a half-wavelength long is an inductive reactance, corresponding to a positive imaginary impedance, whereas if the line were slightly more than $\frac{1}{2}$ wavelength long the input impedance would be a capacitive reactance. Reversal of the angle of the impedance takes place at quarter-wavelength points where the magnitude of the impedance is either zero or infinite.

The statements in this discussion are true for an ideally lossless line. It will be seen in Chap. 4 that when there are small losses the curves of Fig. 3-2 are approached as limiting conditions, but the more extreme values and abrupt discontinuities are moderated.

Voltage and current distribution on a *short-circuited* line are found by making $V_r = 0$ in Eqs. 3-15 and 3-17. This gives

$$V = jI_rZ_0 \sin \beta x \tag{3-21}$$

$$I = I_r \cos \beta x \tag{3-22}$$

The sine curve of voltage distribution and the cosine curve of current distribution are shown in Fig. 3-3*a*. The ratio is

$$Z = \frac{V}{I} = jZ_0 \tan \beta x \tag{3-23}$$

Hence the impedance curve in Fig. 3-3*c* is a tangent curve. The angle of the impedance is +90 degrees for the first quarter-wavelength where the tangent function is positive, and −90 degrees in the second quarter-wavelength where the tangent is negative, and so on. This is shown in Fig. 3-3*b*.

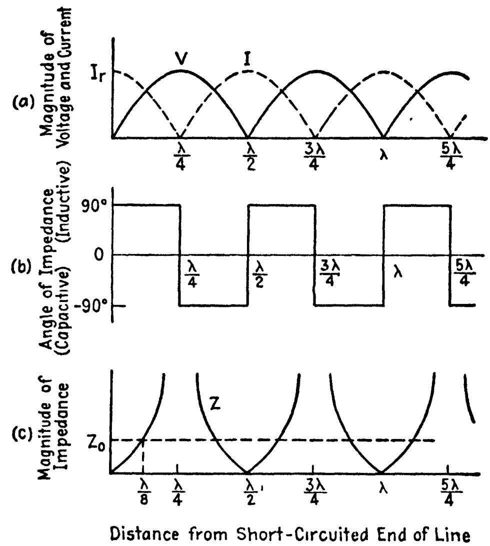


FIG. 3-3. Voltage, current, and impedance on short-circuited line.

Distribution on a line that is terminated in *characteristic impedance* is not as readily deduced from the trigonometric equations. Indeed, when $Z_r = Z_0$ is substituted into Eqs. 3-14 and 3-16, they revert to exponential form:

$$V = V_r(\cos \beta x + j \sin \beta x) = V_r e^{j\beta x} \tag{3-24}$$

$$I = I_r(\cos \beta x + j \sin \beta x) = I_r e^{j\beta x} \tag{3-25}$$

These indicate that the magnitude of V is everywhere that of V_r, and the magnitude of I is everywhere that of I_r, as in Fig. 3-4*a*; and that the angles of both V and I differ from the angles of V_r and I_r by the angle βx. This is shown in Fig. 3-4*b*.

The ratio of voltage to current is

$$Z = \frac{V}{I} = \frac{V_r}{I_r} = Z_0 \tag{3-26}$$

Hence, if the impedance that terminates a line is equal to the characteristic impedance, the input impedance to the line is also equal to the characteristic impedance. This is true for a line of any length, for the input impedance is quite independent of the length of the line. These relations are illustrated in Fig. 3-4*c* and *d*.

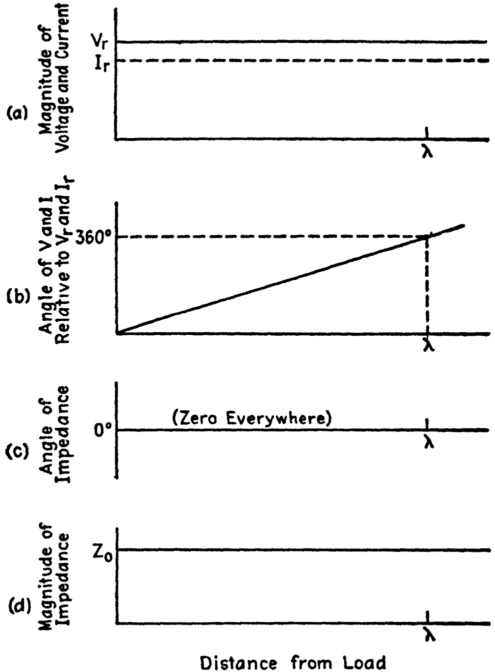


FIG. 3-4. Voltage, current, and impedance on "flat" line.

The curves of Fig. 3-4 for a line terminated in an impedance equal to its characteristic impedance do not have the maxima and minima that appear in Figs. 3-2 and 3-3. Neglecting loss, as we are doing in this chapter, the distribution curves for such a line are merely straight lines. There are no periodic fluctuations owing to resonance along the line. Because of this, a line terminated in its characteristic impedance is, in radio-frequency work, called a "flat" line. The practical advantages of using a "flat" line in communication work have already been mentioned, and will be discussed more fully in Chap. 9.

In Chap. 2 we considered a line with a *load impedance three times the characteristic impedance.* Vector diagrams are shown in Figs. 2-4 and 2-5. For a line so terminated, Eqs. 3-14 and 3-16 can be written

$$V = V_r(\cos \beta x + j\tfrac{1}{3} \sin \beta x) \tag{3-27}$$

$$\begin{aligned} I &= I_r(\cos \beta x + j3 \sin \beta x) \\ &= \frac{V_r}{Z_r}(\cos \beta x + j3 \sin \beta x) \\ &= \frac{V_r}{Z_0}\left(\frac{1}{3} \cos \beta x + j \sin \beta x\right) \end{aligned} \tag{3-28}$$

To see what these equations mean, it would be helpful to show V and I graphically as functions of x, as has been done for lines with other terminations. On this line, however, V and I are complex quantities, as in the above equations.

A complex variable can be plotted as a function of a real variable, but two sets of curves are required. One method of showing V and I as functions of distance is to plot the components of V and I that are in phase with V_r, as in Fig. 3-5*a*, and to plot separately the quadrature components of V and I as in Fig. 3-5*b*. The in-phase and quadrature components are the real and imaginary terms of Eqs. 3-27 and 3-28; they are simply cosine and sine functions as shown.

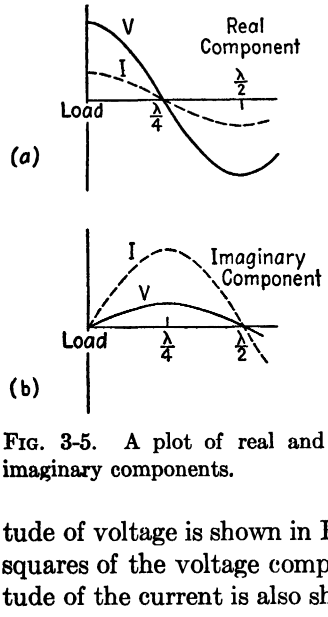


FIG. 3-5. A plot of real and imaginary components.

It is usually more interesting to plot voltage and current by showing amplitudes and angles, however, rather than by showing real and imaginary components. Thus magnitude of voltage is shown in Fig. 3-6*a*; it is the square root of the sum of the squares of the voltage components shown in Fig. 3-5*a* and *b*. The magnitude of the current is also shown, and the corresponding angles are plotted in Fig. 3-6*b*.

Essentially the same information is plotted in Figs. 2-5, 3-5, and 3-6. The first of these cannot show distance along the line graphically. The other two give complete information, but require two sets of curves each. These limitations result from the inherent inadequacy of curves plotted on a plane sheet of paper. If a solid model could be constructed, it would be very satisfactory to use cylindrical coordinates, and to plot distance along an axis, magnitude of voltage (or current) as a radial distance from that axis, and phase angle as an angle about that axis. In this way a single curved line in space would represent magnitude and angle of voltage (or current) at every point of the transmission line. Figure 3-6 would take the form of a lopsided corkscrew. Such a three-dimensional diagram can be imagined more easily than it can be constructed. Figures 2-5, 3-5*a* and *b* can be considered to be the end view, plan, and elevation, respectively, of such a solid model. As a practical matter, however, the plot of magnitude and angle as used in Fig. 3-6 is usually the most satisfactory method of presentation.

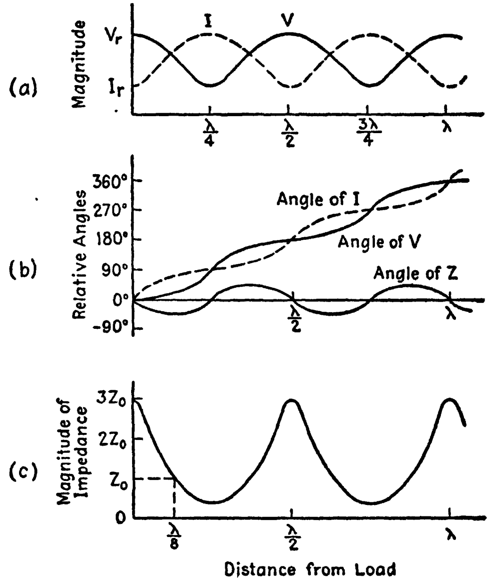


FIG. 3-6. Current, voltage, and impedance, $Z_r = 3Z_0$.

Figure 3-6*c* and *b* show the magnitude and angle of the line impedance V/I. The magnitude of the impedance is something between $3Z_0$ and $\frac{1}{3}Z_0$, depending on the length of line. That is, the input impedance may be as great as the load impedance, or as little as $\frac{1}{9}$ the load impedance.

Although the load is pure resistance, the input impedance to the line may have a phase angle as great as 53 degrees. The angle of Z is the angle of V minus the angle of I. Input impedance to the line has a capacitive reactance in the first quarter-wavelength, an inductive reactance in the second quarter-wavelength, and continues to alternate in successive quarter-wavelengths.

It is evident that the curves of Fig. 3-6 are intermediate between the curves of Fig. 3-2 and those of Fig. 3-4. It is natural to find that the behavior of a line terminated in high resistance ($Z_r = 3Z_0$) is between that of an open-circuited line ($Z_r = \infty$) and that of a *flat* line ($Z_r = Z_0$).

For instance, the voltage does not become zero at the odd quarter-wavelength points, as it does on an open-circuited line, but the minimum voltage is one-third of the maximum. Similarly, the maximum current is three times the minimum. For this reason the line is said to have a *standing-wave ratio* of 3.

4. Variable Frequency. Curves in this chapter (Figs. 3-2 to 3-6) show voltage, current, and impedance as functions of the variable βx. We have taken the independent variable to be distance x along the line, assuming frequency to be constant, and have indicated distances in terms of the wavelength λ.

All the curves of these figures are just as satisfactorily interpreted by assuming that we are discussing a line of fixed length and changing the frequency. The independent variable is then β, the product βx varies, and x remains constant; $\beta = \omega\sqrt{lc}$, and is hence proportional to frequency. When so interpreted, the curves give sending-end voltage, current, and impedance as functions of applied frequency, and the calibration of the horizontal axis is in terms of the number of wavelengths in the physical length of the line.

Interpreted in this way, Fig. 3-6 indicates that the input impedance to a line for which $Z_r = 3Z_0$ is maximum when the frequency is such that the length of the line is $\frac{1}{2}\lambda$. It is again maximum if the frequency is increased to make the wavelength equal to the length of the line. If the wavelength is four-thirds the length of the line ($x = \frac{3}{4}\lambda$), input impedance is minimum, and so on.

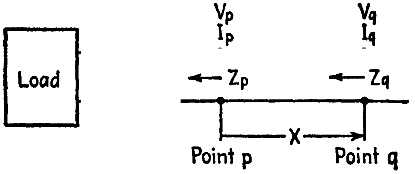


FIG. 3-7. Arbitrary points on a transmission line.

5. Other Trigonometric Equations. Equations 3-14 to 3-17 express voltage and current at a point on a transmission line in terms of the receiving-end voltage and current and the distance of the point from the receiving end. Such equations can just as well be written to give voltage and current at any point q (see Fig. 3-7) in terms of the voltage and current at any other point p and the distance x from point p to point q:

$$V_q = V_p \cos \beta x + jI_pZ_0 \sin \beta x \tag{3-29}$$

$$I_q = I_p \cos \beta x + j\frac{V_p}{Z_0} \sin \beta x \tag{3-30}$$

The distance x is measured from p to q in the positive direction, which is defined as being away from the load and toward the generator.

To see that these equations are valid, it is only necessary to review the derivations of Eqs. 3-14 and 3-17, for it will be found that there is nothing unique about the point referred to as the receiving end of the line, and this reference point may actually be any point. Or, from a slightly different point of view, we may consider that everything to the left of point p in Fig. 3-7 constitutes the load on the section of transmission line that extends from point q to point p; and that point p is indeed the receiving end of this section of line. The impedance "looking toward the load" at point p is Z_p; this is the receiving-end impedance of the section of line under consideration. Z_q, the impedance looking toward the left at point q, is the sending-end impedance of the section of line from q to p.

Figure 3-7 shows point q at a greater distance from the load than is point p. Thus x is a positive quantity. However, Eqs. 3-29 and 3-30 can be used even though point q is between point p and the load; in such a case, the distance x is negative. To avoid using negative distances it is customary to let some other symbol, say u, represent distance measured toward the receiving end. Obviously, $u = -x$. Then

$$V_q = V_p \cos \beta u - jI_pZ_0 \sin \beta u \tag{3-31}$$

$$I_q = I_p \cos \beta u - j\frac{V_p}{Z_0} \sin \beta u \tag{3-32}$$

A special form of these equations gives voltage and current at any point in terms of *sending-end* conditions:

$$V = V_s \cos \beta u - jI_sZ_0 \sin \beta u \tag{3-33}$$

$$I = I_s \cos \beta u - j\frac{V_s}{Z_0} \sin \beta u \tag{3-34}$$

This pair of equations is less often useful than are Eqs. 3-14 to 3-17 because the sending-end current is usually not known, and has to be computed from the receiving-end impedance. There are occasional applications, however, to lines for which sending-end current is specified and receiving-end impedance is to be found. It has previously been emphasized that computations on a transmission line must ordinarily start at the receiving end of the line, and the difficulty of using Eqs. 3-33 and 3-34 provides another illustration of that fact.

6. Line Impedance. We have considered the impedance of transmission lines with certain terminations. Now, having general equations for voltage and current, a general trigonometric expression for impedance is easily found. Taking impedance to be the ratio of voltage to current at any point on the transmission line, as in Eqs. 2-34 and 3-20, the impedance is found by dividing Eq. 3-14 by 3-16:

$$Z = \frac{V}{I} = \frac{V_r\left(\cos\beta x + j\frac{Z_0}{Z_r}\sin\beta x\right)}{I_r\left(\cos\beta x + j\frac{Z_r}{Z_0}\sin\beta x\right)}$$

$$= Z_0\frac{Z_r\cos\beta x + jZ_0\sin\beta x}{Z_0\cos\beta x + jZ_r\sin\beta x} \tag{3-35}$$

The form of equation giving *relative* impedance is often more useful:

$$\frac{Z}{Z_0} = \frac{\frac{Z_r}{Z_0}\cos\beta x + j\sin\beta x}{\cos\beta x + j\frac{Z_r}{Z_0}\sin\beta x} \tag{3-36}$$

With certain line terminations, Eqs. 3-35 and 3-36 reduce to the simple forms that have already been discussed. For a short-circuited line, $Z_r = 0$ and Eq. 3-35 becomes

$$Z = jZ_0\tan\beta x \tag{3-37}$$

For an open-circuited line, $Z_r = \infty$ and Eq. 3-35 reduces to

$$Z = -jZ_0\cot\beta x \tag{3-38}$$

For a "flat" line, $Z_r = Z_0$ and Eq. 3-35 gives

$$Z = Z_0 \tag{3-39}$$

7. Trigonometric Functions. When the transmission-line equations are written as trigonometric functions, the argument of the function is such a quantity as βx. The quantity βx is a dimensionless number. Though we wish to find its sine, cosine, or tangent, it is not an angle.

It is customary in elementary trigonometry to give geometrical definitions of the sine, the cosine, and the tangent. They are defined as the ratios of sides of triangles. This geometrical definition is satisfactory for elementary purposes, but is severely limited. For more general application, it is best to take a broader definition of the functions, and to consider the geometrical use to be a special interpretation.

Let us define the sine and cosine in terms of the exponential function

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{3-40}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{3-41}$$

The exponential function, in turn, is defined by the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{3-42}$$

Sine and cosine being defined, tangent is defined as the ratio

$$\tan x = \frac{\sin x}{\cos x} \tag{3-43}$$

These are the first four formulas of Table II (inside back cover).

By combining these definitions, the trigonometric functions can be expressed, if desired, as infinite series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3-44}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{3-45}$$

In all these expressions, x can be any number. It has no necessary relation to angle, and it is certainly not to be conceived as an angle measured in degrees. It is possible for x to have any value, positive or negative, and x can be imaginary or complex as well as real; all these possibilities lie within the scope of the above definitions.

Numerical values of the sine, cosine, and tangent are most conveniently found, for transmission-line work, from tables prepared with radian arguments. The radian, so-called, is really a dimensionless pure number, and need not be geometrically interpreted as an angle. Since tables of trigonometric functions of numbers (or radians) are relatively rare, such a table is given in this book as Appendix 1. Reference 31 is a much more extensive table.

If tables of radian arguments are not available, it is possible to use a table prepared in degrees: the argument, really a pure number, is assumed to be in radians, and the equivalent number of degrees is computed, and the table is then entered with this value. For instance, to find sin βx when $\beta x = 1.00$: the desired value is the same as the sine of 1.00 radian, or (approximately) of 57.3 degrees. The sine of 57.3 degrees is then found from an ordinary table to be 0.8415, which is therefore sin 1.00.

Occasionally, if tables are inadequate, or for special purposes, it is helpful to compute a function from the infinite series. Thus, to find sin 1.00, using Eq. 3-44,

$$\begin{aligned} \sin 1.00 &= 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5{,}040} + \cdots \\ &= 1 - 0.16667 + 0.00833 - 0.00020 + \cdots \\ &= 0.8415 \end{aligned} \tag{3-46}$$

which is the same as the value obtained from a table in degrees, and may also be compared with the table of Appendix 1.

8. Applications. In the first section of this chapter it was mentioned that there are many practical problems for which r and g, the resistance of the wires and conductance of the insulation, can be entirely neglected. This is possible only if r is quite small compared with ωl and g is quite small compared with ωc.

The trigonometric formulas and the simple standing-wave pictures of current and voltage distribution are characteristic of lines operating at very high frequency, in the range of hundreds or thousands of megacycles per second. The operation of such lines, and the application of the equations of this chapter, will be discussed in Chap. 12.

For the simple equations of this chapter to apply, however, although the above requirement is necessary, it is not sufficient. It is also necessary that the total loss in the whole length of the line be small. Most radio-frequency lines satisfy this requirement also, for whereas power lines and telephone lines are commonly several miles or even hundreds of miles long, most radio-frequency lines are to be measured in meters or centimeters. This is not always true, however. Long coaxial lines are obvious exceptions, and it can by no means be assumed that the output power of such a long line is equal to its input power; if that assumption cannot be made, the equations of this chapter do not apply.

Still another precaution is necessary in using the simple trigonometric equations of this chapter. They may give excellent approximations of voltage, current, and impedance at almost all points along a transmission line, but exactly at a node, where they predict exactly zero voltage or zero current, and either zero or infinite impedance, they always fail completely. However small the current or voltage may be at a node, it cannot be quite zero. An equation that gives a value of zero at a node is therefore in serious error at that point. Since values at the nodal points are of the greatest practical importance, they will require careful attention in Chap. 12. It was to problems of this kind that item 2 of the classification on page 37 referred.

Most telephone lines and power lines, and even some radio-frequency lines have so much loss that the approximations of this chapter cannot be applied. For such lines, it is necessary to revise the equations of voltage and current distribution, and impedance, to include the effect of line loss. Fortunately, this can be done rather easily, and without greatly changing the form of the equations, as will be seen in the next chapter.

PROBLEMS

3-1. If loss is negligible and $K_\varphi = 0$, find the velocity of propagation on a two-wire line in meters per second. Use Formulas 3 and 6 of Table I.

3-2. If loss is negligible and $K_\varphi = 0$, find the velocity of propagation on a coaxial line, using Formulas 7 and 8 of Table I.

3-3. Derive Eqs. 3-16 and 17 for current distribution from Eq. 3-7.

3-4. Draw curves to show the combination of two traveling sine waves into a standing wave as in Fig. 3-1*c*, but continue the diagram until one complete cycle has been illustrated.

3-5. Draw curves to show the combination of two traveling sine waves, as in Fig. 3-1*c*, but let the two traveling waves have different amplitudes. What is the nature of the result?

3-6. The receiving-end voltage of an open-circuited 60-cycle power line is 127,000 volts (to neutral). The line is $\frac{1}{10}$ wavelength long. Neglect line losses. What is the sending-end voltage? How many miles long is the line?

3-7. If the sending-end voltage of the line of Prob. 3-6 is 127,000 volts, what is the receiving-end voltage?

3-8. If the characteristic impedance of the line of Prob. 3-6 is 390 ohms, what is the sending-end current? (Receiving-end voltage is 127 kilovolts.)

3-9. What is the sending-end impedance of the open-circuited line of Prob. 3-6?

3-10. What is the sending-end impedance of a lossless line open-circuited at the receiving end, the line being 9.0 centimeters long, at an operating frequency of 1,000 megacycles per second? Characteristic impedance of the line is 70 ohms.

3-11. What is the sending-end impedance of the line of Prob. 3-10 if the line is short-circuited at the receiving end?

3-12. A line leading to a broadcast transmitter antenna operates at a frequency of 1,200 kilocycles. It is 100 meters long, and has a 500-ohm characteristic impedance. What is the input impedance if the remote end of the line is (*a*) open-circuited? (*b*) short-circuited? Neglect loss.

3-13. $Z_r = 2Z_0$. Compute V/V_r and IZ_0/V_r, and plot as in Fig. 2-5. Compute the magnitude and angle of Z/Z_0, and plot as in Fig. 3-6.

3-14. Compute and plot the values required in Prob. 3-13, but for a line on which $Z_r = \frac{1}{2}Z_0$.

3-15. Determine sin 0.5 and sin 2.0 by the following methods (obtain results correct to four figures):

a. Compute from infinite series.

b. Read from the table of Appendix 1.

c. Read from a table using degrees.

3-16. Determine cos 0.5 and cos 2.0 by the three methods of Prob. 3-15 (obtain results correct to four figures).

CHAPTER 4

LINES WITH LOSS

1. Voltage and Current Equations. It was shown in Chap. 3 that voltage and current distribution on a transmission line, and the line impedance, can conveniently be given mathematical expression in the form of trigonometric functions if the frequency is so high, or the line loss so low, that losses can be neglected. The trigonometric functions describe standing waves on the line, standing waves that result from the superposition of two traveling waves moving in opposite directions.

The trigonometric equations were derived from the exponential equations. The general exponential expression for voltage distribution is

$$V = V_1 e^{\gamma x} + V_2 e^{-\gamma x} \tag{4-1}$$

and when loss is zero this can be written (as in Eq. 3-6)

$$V = V_1 e^{j\beta x} + V_2 e^{-j\beta x} \tag{4-2}$$

which, in Sec. 2, Chap. 3, was transformed to

$$V = V_r\left(\cos \beta x + j\frac{Z_0}{Z_r}\sin \beta x\right) \tag{4-3}$$

A comparable form will now be derived without making the assumption that loss is negligible. Equation 4-1, a solution of the differential equations of the transmission line that includes the effect of line loss, provides a starting point for the new derivation. For convenience, let us introduce a new symbol B that is defined by its relation to γ in the following equation:

$$\gamma = jB \tag{4-4}$$

When this new symbol is introduced into Eq. 4-1, we obtain

$$V = V_1 e^{jBx} + V_2 e^{-jBx} \tag{4-5}$$

and the similarity of Eq. 4-5 to Eq. 4-2 is obvious. Steps paralleling those of Chap. 3 may now be taken to put Eq. 4-5 into trigonometric form.

Equation 4-5 is modified by substituting into it Eqs. 2-27 for V_1 and 2-28 for V_2. The result is

$$V = V_r \frac{Z_r + Z_0}{2Z_r} e^{jBx} + V_r \frac{Z_r - Z_0}{2Z_r} e^{-jBx} \tag{4-6}$$

Regrouping terms

$$V = V_r\left(\frac{e^{jBx} + e^{-jBx}}{2} + j\frac{Z_0}{Z_r}\frac{e^{jBx} - e^{-jBx}}{2j}\right) \tag{4-7}$$

The first of these fractions is identical with cos Bx. The second fraction is equivalent to sin Bx. Hence

$$V = V_r\left(\cos Bx + j\frac{Z_0}{Z_r}\sin Bx\right) \tag{4-8}$$

or, since $V_r/Z_r = I_r$,

$$V = V_r \cos Bx + jI_rZ_0 \sin Bx \tag{4-9}$$

Equation 4-8 includes the effect of line loss, whereas Eq. 4-3 does not. The difference is in the use of B rather than β. B is a complex quantity. From its definition in Eq. 4-4 it is seen that

$$jB = \gamma \doteq \alpha + j\beta \tag{4-10}$$

from which

$$B = \frac{\gamma}{j} = -j\gamma = \beta - j\alpha \tag{4-11}$$

It is apparent that if a transmission line has negligible loss, so that the attenuation constant α is zero, B is identical with β. In this ideal case, B is a purely real quantity. When line loss is considered, B has an imaginary component equal to $-j\alpha$. On most practical lines, losses are small enough so that α is relatively small and the quantity B is not greatly different from the quantity β.

Equation 4-8 or 4-9 gives voltage distribution along a line. Current distribution is given by similar equations, derived in the same way:

$$I = I_r\left(\cos Bx + j\frac{Z_r}{Z_0}\sin Bx\right) \tag{4-12}$$

$$I = I_r \cos Bx + j\frac{V_r}{Z_0}\sin Bx \tag{4-13}$$

2. Trigonometric Functions of Complex Variables. These equations for voltage and current distribution that take line loss into account have a familiar appearance, but they include the sine and cosine of the complex variable Bx. To make use of these equations, it is necessary for trigonometric functions of a complex variable to be understood.

The definitions that we already have adopted for the sine, cosine, and tangent are broad enough to include complex arguments, and the possibility of using complex arguments was mentioned in connection with the definitions on page 50. The sine and cosine are given in Eqs. 3-44 and 3-45 as

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{4-14}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \tag{4-15}$$

(These expressions are given also in Table II.) Using these definitions (which we do, consistently, throughout this book) z can be real, or it can be complex or imaginary.[1]

It is possible to use the series of Eqs. 4-14 and 4-15 for actual computation to determine the numerical values of the functions of a complex variable, for only the familiar operations of multiplication and addition of complex quantities are required to evaluate the terms. Enough terms can be used to give any desired precision. The computation is likely to be tedious, however, unless z is small, and other methods of evaluation are usually preferable.

The trigonometric functions can be evaluated from their exponential equivalents. Referring to Formula 3 of Table II for example, we may use

$$\cos z = \frac{e^{jz} + e^{-jz}}{2} \tag{4-16}$$

The exponentials in this formula can be evaluated even though z is complex, for if $z = x + jy$

$$e^{jz} = e^{jx-y} = e^{-y}e^{jx} = e^{-y} \angle x \tag{4-17}$$

This is handled like any complex quantity, the value of e^{-y} being found from a table of exponentials.

However, the sine and cosine of a complex argument are often most easily found by expanding the complex functions into expressions that contain only functions of real variables. This method requires the use of hyperbolic functions, and since hyperbolic functions are not as generally familiar as trigonometric functions, it will be well to review the meaning of hyperbolic functions of a real variable before proceeding with the discussion of functions of a complex variable.

Real Hyperbolic Functions. The hyperbolic functions *sinh* and *cosh* are closely related to the circular functions *sin* and *cos*. The hyperbolic sine and hyperbolic cosine are defined in Table II as

---

[1] The use of z in these equations does not imply any connection with impedance. In these abstract equations, we follow the mathematical custom of using z as a complex variable and x and y as real variables. Also w will be used as another complex variable, and u and v as real variables.

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \tag{4-18}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \tag{4-19}$$

The hyperbolic tangent is defined as the ratio

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{4-20}$$

The hyperbolic sine, cosine, and tangent of a *real* variable are plotted in Fig. 4-1, together with the exponential function to which they are closely related. Unlike the circular functions, they are not cyclic. Some of their more useful characteristics are as follows.

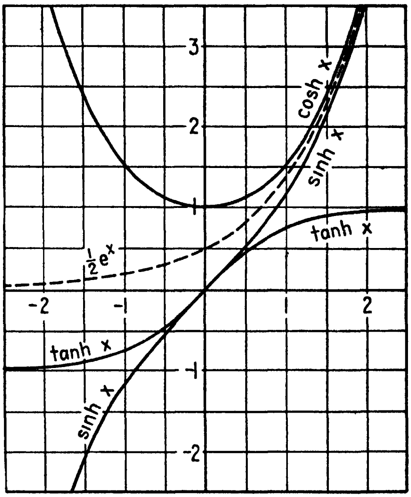


FIG. 4-1. Real hyperbolic functions.

cosh x, with x real, is never less than 1, and is positive for both positive and negative values of x. For very small values of x, cosh x approaches 1, and its rate of change approaches zero. For very large values of x, cosh x approaches $\frac{1}{2}e^x$; this is most clearly evident from the equation

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{4-21}$$

for if x is large, e^{-x} is vanishingly small.[2]

The sinh of a real variable may have any value from $-\infty$ to $+\infty$. If $x = 0$, $\sinh x = 0$. For small values of x, sinh x approaches x, and its curve approximates a straight line at 45 degrees through the origin. For very large positive values of x, sinh x, like cosh x, approaches $\frac{1}{2}e^x$. If x is negative, sinh x is negative. The hyperbolic sine, like the circular sine, is an *odd* function, whereas the hyperbolic cosine and circular cosine are *even* functions.

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[2] It is convenient, if somewhat inelegant, to pronounce the abbreviations *cosh* and *tanh* to rhyme with gauche and ranch. *sinh* is not so easily pronounced, but is sometimes spoken as if it were spelled cinch or shine, or, perhaps most commonly, as sīnch, with long i.

tanh x is defined as sinh x divided by cosh x. Hence it is zero if x is zero, and its curve, like that of the sinh function, approximates a straight line at 45 degrees as it passes through the origin. If x is real, tanh x must lie between -1 and $+1$, and these are the limits of the function as x approaches $-\infty$ and $+\infty$.

Various useful formulas relating to trigonometric and hyperbolic functions are collected in Table II. All these formulas can be derived from the definitions in the first seven equations of Table II. The formulas relating to trigonometric functions will be familiar to most readers, and the formulas for hyperbolic functions are quite similar.

Formulas 14 to 19 of the table show relations between circular and hyperbolic functions. For instance, Formula 14 states that $\sin jy = j \sinh y$. To verify this expression, substitute jy for z in Formula 2 of Table II which defines the sine, and substitute y for z in Formula 5 which defines the hyperbolic sine. Either the exponential or the series form of Formulas 2 and 5 may be used, and in either case the relation of Formula 14 of the table is readily obtained.

The other formulas of this group in Table II are derived in a similar manner. It is interesting to notice that the sine of an imaginary quantity is imaginary, but the cosine of an imaginary quantity is real. This group of equations will be immediately useful in expanding the trigonometric functions of a complex variable into a form that contains only functions of real variables.

Expansion of the Complex Functions. Usually the value of the sine or cosine of a complex quantity is most easily found by rewriting the function as follows. Assume that the complex variable z has the real and imaginary components x and jy:

$$z = x + jy \tag{4-22}$$

Taking the cosine function as an example, it may be written

$$\cos z = \cos (x + jy) \tag{4-23}$$

This can now be expanded by using the formula for the cosine of the sum of two quantities (Formula 21 of Table II):

$$\cos z = \cos x \cos jy - \sin x \sin jy \tag{4-24}$$

Formulas 14 and 15 of the table are now used, to give

$$\cos z = \cos x \cosh y - j \sin x \sinh y \tag{4-25}$$

Since both x and y are real, each term in this expansion can be evaluated by reference to almost any handbook,[3] and a numerical value of the cosine of a complex variable z is thereby obtained.

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[3] For convenience, short tables of circular, hyperbolic, and exponential functions are included in the appendixes.

The similar expansion of the sine function is

$$\begin{aligned}\sin z &= \sin (x + jy)\\ &= \sin x \cosh y + j \cos x \sinh y \qquad (4\text{-}26)\end{aligned}$$

In applying these relations to a transmission line, as in Eq. 4-8 or 4-12, the complex argument is usually $Bx = \beta x - j\alpha x$, the x in this equation being distance. The sine and cosine of Bx are expanded by using the general forms of Eqs. 4-25 and 4-26 to give

$$\begin{aligned}\sin Bx &= \sin (\beta x - j\alpha x)\\ &= \sin \beta x \cosh \alpha x - j \cos \beta x \sinh \alpha x \qquad (4\text{-}27)\\ \cos Bx &= \cos (\beta x - j\alpha x)\\ &= \cos \beta x \cosh \alpha x + j \sin \beta x \sinh \alpha x \qquad (4\text{-}28)\end{aligned}$$

Finally, these expressions can be substituted into the transmission-line equations. Equation 4-8, for example, which gives voltage distribution on a line with loss, can be written entirely in terms of real functions, giving

$$\begin{aligned}V = V_r\Big[&\cos \beta x\left(\cosh \alpha x + \frac{Z_0}{Z_r} \sinh \alpha x\right)\\ &+ j \sin \beta x\left(\sinh \alpha x + \frac{Z_0}{Z_r} \cosh \alpha x\right)\Big] \qquad (4\text{-}29)\end{aligned}$$

This form of the voltage equation can be readily evaluated from ordinary tables, and, indeed, the equation will always take some form equivalent to this if ordinary tables of real variables are to be used for computation.

3. Hyperbolic Functions of Complex Variables. The solution of the differential equations of the transmission line was first expressed in *exponential* form, as Eqs. 1-26 and 1-27. The solution was then rewritten in *trigonometric* form, in Eqs. 4-8 and 4-12, for greater convenience of computation. The solution can also be expressed in the form of complex *hyperbolic* functions. Indeed, hyperbolic functions are quite commonly used for this purpose, probably for historical reasons.

Starting with the exponential solution for voltage, Eq. 4-1, we may substitute into it Eqs. 2-27 and 2-28, giving

$$V = V_r\left(\frac{e^{\gamma x} + e^{-\gamma x}}{2} + \frac{Z_0}{Z_r}\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right) \qquad (4\text{-}30)$$

These fractions may be recognized as the hyperbolic cosine and hyperbolic sine functions, and the equation can be written

$$V = V_r\left(\cosh \gamma x + \frac{Z_0}{Z_r} \sinh \gamma x\right) \qquad (4\text{-}31)$$

or

$$V = V_r \cosh \gamma x + Z_0 I_r \sinh \gamma x \qquad (4\text{-}32)$$

Similarly,

$$I = I_r\left(\cosh \gamma x + \frac{Z_r}{Z_0} \sinh \gamma x\right) \tag{4-33}$$

or

$$I = I_r \cosh \gamma x + \frac{V_r}{Z_0} \sinh \gamma x \tag{4-34}$$

On an ideal line, or a high-frequency line on which loss can be neglected, γ is purely imaginary. Equations 4-31 to 4-34 can then be reduced to the circular trigonometric functions of a real variable, as in Chap. 3. But if line loss is to be considered in the above equations, hyperbolic functions of complex variables must be evaluated.

Methods of evaluating the complex hyperbolic functions are similar to those discussed for the circular trigonometric functions:

1. From infinite series, Eqs. 4-18 and 4-19, or Formulas 5 and 6 of Table II, using as many terms as needed.
2. From exponential expansions, as in Formulas 5 and 6 of Table II.
3. From trigonometric expansions, as in Formulas 26 and 27 of Table II.

The latter equations, which are

$$\sinh (x + jy) = \sinh x \cos y + j \cosh x \sin y \tag{4-35}$$

$$\cosh (x + jy) = \cosh x \cos y + j \sinh x \sin y \tag{4-36}$$

provide the most convenient means of computation for most purposes.

The argument of the complex hyperbolic functions appearing in Eqs. 4-31 to 4-34 is $\gamma x = \alpha x + j\beta x$. The hyperbolic sine and cosine of γx are

$$\begin{aligned} \sinh \gamma x &= \sinh (\alpha x + j\beta x) \\ &= \sinh \alpha x \cos \beta x + j \cosh \alpha x \sin \beta x \end{aligned} \tag{4-37}$$

$$\begin{aligned} \cosh \gamma x &= \cosh (\alpha x + j\beta x) \\ &= \cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x \end{aligned} \tag{4-38}$$

By introducing Eqs. 4-37 and 4-38, the complex hyperbolic functions of Eqs. 4-31 to 4-34 can be replaced. When this is done, the equations for voltage and current are written in terms of functions of real variables, and evaluation may proceed by use of tables found in ordinary reference books. For example, Eq. 4-31 for voltage distribution can be expanded to give

$$\begin{aligned} V = V_r\Big[& \cos \beta x\left(\cosh \alpha x + \frac{Z_0}{Z_r} \sinh \alpha x\right) \\ & + j \sin \beta x\left(\sinh \alpha x + \frac{Z_0}{Z_r} \cosh \alpha x\right)\Big] \end{aligned} \tag{4-39}$$

and in this form, which is the necessary form for computation, we find that we have reached exactly the same expression that was obtained as Eq. 4-29 by expanding the complex *circular* functions.

Actual computation of transmission-line quantities is therefore the same whether the general form of an equation is written in circular or hyperbolic functions. It is optional with the reader whether he prefers to use circular or hyperbolic functions, for all equations may be expressed in either form. The hyperbolic functions of a complex quantity are not fundamentally more difficult than the circular functions. It is only because they are less familiar that they seem so.

4. Charts of Complex Functions. When values of the complex functions have been computed, they can be preserved, for later use, in a table or a set of curves. It is easy to tabulate or plot the circular and hyperbolic functions of a *real* variable, as in Fig. 4-1, but to plot a function of a *complex* variable is more difficult. It is necessary to indicate both real and imaginary components of the variable and both real and imaginary components of the function. A four-dimensional plot would be convenient, but lacking this it is possible to use a two-dimensional plot with two families of curves. The plot may be arranged in any of several ways.

For instance, if it is desired to plot the cosine of a complex variable, which will itself be a complex quantity, we may write

$$\cos Me^{i\mu} = Se^{i\sigma} \tag{4-40}$$

and then plot S and σ as functions of M and μ. Note that M, μ, S, and σ are real. M is the magnitude of the quantity of which the cosine is to be determined, and μ is its angle.

If we were finding the cosine of a real number, μ would be zero and we would find merely cos M. If the angle μ is small, it is to be expected that cos $Me^{i\mu}$ will not differ much from cos M. This is true, and is shown in Fig. 4-2*a*. In that figure the magnitude S of the cosine function is plotted; the lowest curve is for a real variable ($\mu = 0$) and is hence the familiar cosine curve. The next curve above it shows the magnitude of the cosine for a variable having an angle (with the axis of reals in the complex plane) of 5 degrees. (This same curve gives the magnitude of the cosine function if the angle of the argument is -5 degrees.)

The next curve is for an argument with a ±10-degree angle, and other successive curves are for variables with increasing angles until, with an angle of ±90 degrees, the argument is purely imaginary. Since $\cos jx = \cosh x$, this last curve is a plot of the hyperbolic cosine of a real variable, and is the same as the hyperbolic cosine curve in Fig. 4-1.

Expressed in other words: The curve for a real variable is the ordinary cosine curve, the curve for an imaginary variable is the hyperbolic cosine curve, and the curves for complex variables lie between those two.

The cosine of a complex quantity, however, is itself complex, and cannot be expressed merely as a magnitude. We must know either its real and imaginary components or else its magnitude and angle. The latter form is used in Fig. 4-2; the magnitude is shown in Fig. 4-2*a*, and the angle in Fig. 4-2*b*.

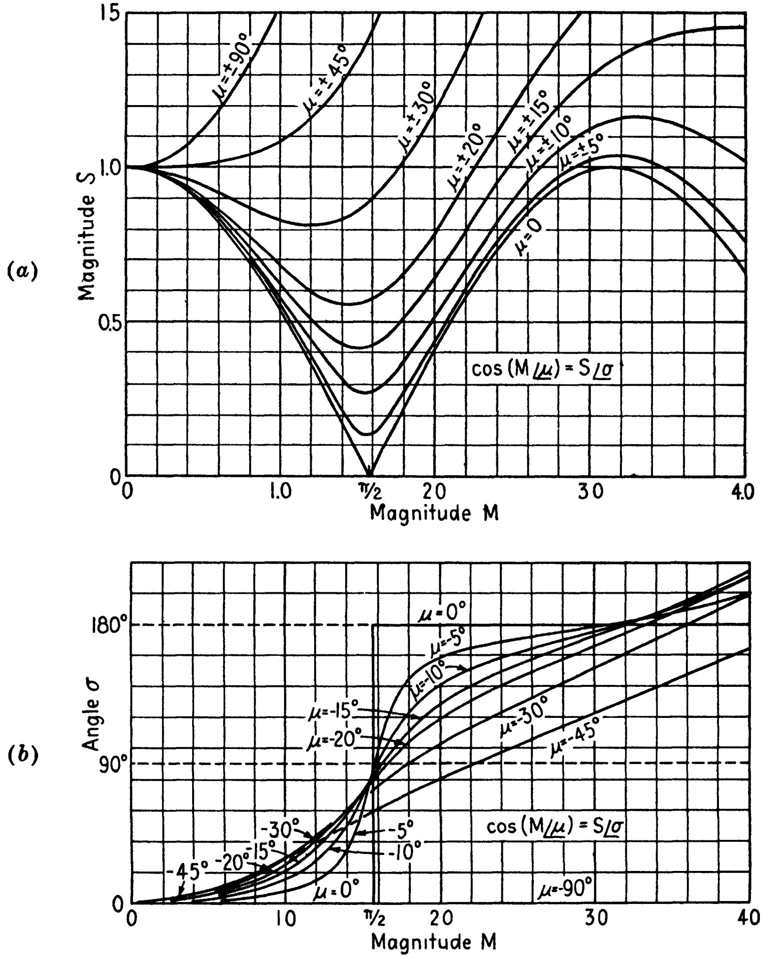

FIG. 4-2. Chart of $\cos Me^{i\mu} = Se^{i\sigma}$ (μ and σ in degrees).

When the variable is real ($\mu = 0$), the angle of the function is zero until $M = \pi/2$, when the angle suddenly becomes 180 degrees. (This abrupt change corresponds to the change of sign of the function; the cosine of a real variable changes from positive to negative as the argument passes $\pi/2$.) If μ has a small value (let us say -5 degrees), the angle σ of the cosine function rises quickly, but without discontinuity, when M approaches and passes $\pi/2$. For small values of M, σ is almost zero; if M is near π, σ is almost 180 degrees. The discontinuous curve for the real variable is approached as a limit if the angle μ of the argument becomes very small.

Other curves for arguments with increasing (negative) angle are shown. Finally, when the argument is purely imaginary ($\mu = -90$ degrees), the function is purely real ($\sigma = 0$). This, again, is in agreement with the relation $\cos jx = \cosh x$. Curves for complex arguments approach more nearly the curve for a real variable or the curve for an imaginary variable, depending on whether the argument is more nearly real or imaginary.

Although the magnitude S of the cosine is the same for either positive or negative values of μ, the sign of σ is dependent upon the sign of μ. This can be seen from Eq. 4-25. If $Me^{j\mu} = x + jy$,

$$\begin{aligned}\cos Me^{j\mu} &= \cos (x + jy) \\ &= \cos x \cosh y - j \sin x \sinh y = Se^{j\sigma} \qquad (4\text{-}41)\end{aligned}$$

Reversing the sign of μ will reverse the sign of y. Reversal of the sign of y will not change $\cosh y$, but it will reverse the sign of $\sinh y$ (see Table II) and therefore of the imaginary component of $Se^{j\sigma}$. This results in reversal of the sign of the angle σ. In mathematical terms, the cosine of the conjugate of a quantity is equal to the conjugate of the cosine of that quantity.[4]

The curves of Fig. 4-2 are drawn for negative values of μ in order to include the range that is useful for transmission-line equations. In Eqs. 4-8 and 4-12 for instance, Bx is a quantity with a negative imaginary component, for $B = \beta - j\alpha$, and hence, when written in polar form, μ is negative. See also Eqs. 4-27 and 4-28.

Because of the conjugate relationship, the same curves could be used for positive values of μ, instead of negative, by taking σ to be negative instead of positive. That is, the signs of σ and μ in Fig. 4-2*b* could both be reversed.

A similar set of curves for the sine function of a complex variable is shown in Fig. 4-3. For a real variable, with $\mu = 0$, the familiar sine curve appears in Fig. 4-3*a*. For an imaginary variable, with $\mu = \pm 90$

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[4] It is readily shown that not only does $\cos \bar{z} = \overline{\cos z}$ (using the bar above a quantity to indicate the conjugate of that quantity, in the usual mathematical notation, so that if $z = x + jy$, $\bar{z} = x - jy$), but also $\sin \bar{z} = \overline{\sin z}$. Indeed, there is a theorem that if R is any rational operation,

$$R(\bar{a}) = \overline{R(a)} \qquad (4\text{-}42)$$

and this includes the trigonometric functions which can be expanded into series of rational terms.

degrees, the magnitude is that of the hyperbolic sine and the curve is similar to the hyperbolic sine curve of Fig. 4-1. If the variable has some complex value, the magnitude of the sine lies between these extremes, as shown.

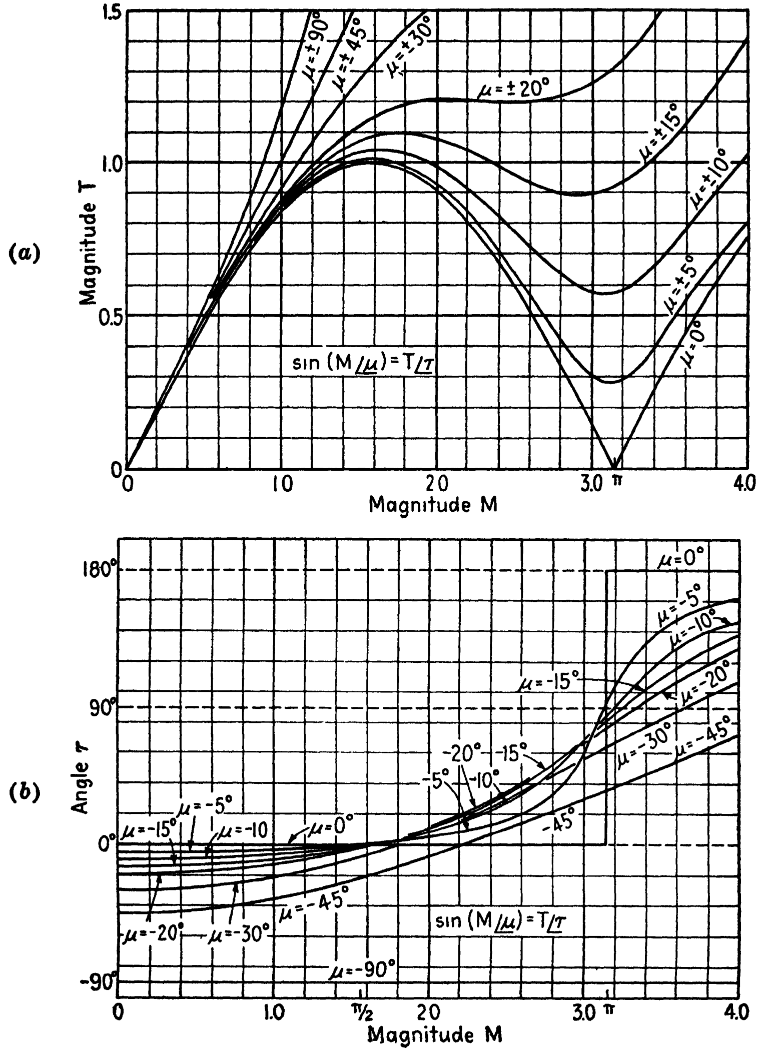

FIG. 4-3. Chart of sin $Me^{j\mu} = Te^{j\tau}$ (μ and τ in degrees).

The angle of the sine, called τ, is indicated in Fig. 4-3*b*. Unlike the cosine, the sine function is imaginary if the argument is imaginary. Like the cosine function, however, if the sign of μ is reversed the sign of τ is reversed also (see footnote 4).

Figures 4-2 and 4-3 show how the complex sine and cosine are related to the more familiar sine and cosine of a real variable, but there are other ways of plotting the functions of complex arguments that are more compact and better for use in computation.[5] One such plot is presented in Fig. 4-4. Let us consider any complex quantity z, with components x and jy,

$$z = x + jy \tag{4-43}$$

Figure 4-4 is a plot of the z plane, with two families of curves superimposed to show the magnitude and angle of sin z. Any point in the chart

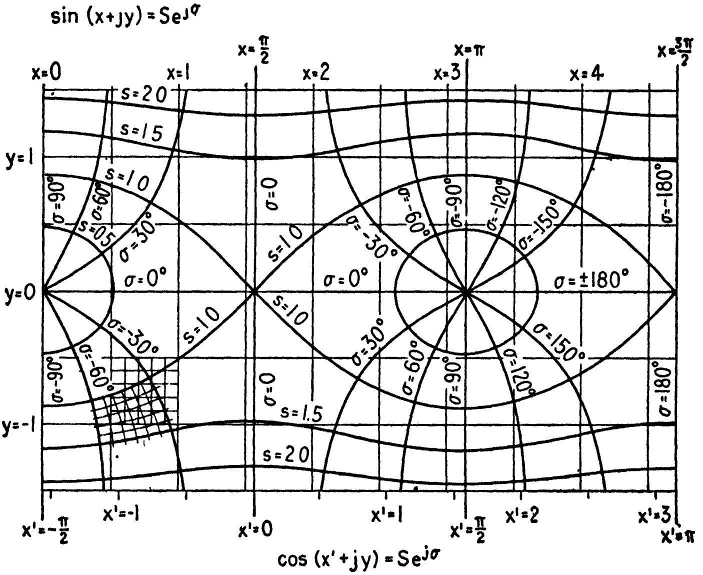

FIG. 4-4. Chart of complex sine and cosine (σ in degrees), $x' = x - \pi/2$.

represents a value of z by reference to the rectangular coordinate system; vertical lines give values of x, using the scale at the *top*, and horizontal lines give values of y, referring to the scale in the left margin. Each point represents the corresponding value of sin z by reference to the superimposed coordinate system of curves. sin z is a complex quantity, and it is read from this chart in the polar form $Se^{j\sigma}$. The more or less elliptical lines give values of the magnitude S, and the more or less radial lines give values of the angle σ.

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[5] The reader who does not wish to use charts for computation may omit the rest of this chapter.

To find sin $(x + jy)$ from the chart, first locate a point with coordinates x and y, referring to vertical and horizontal straight lines. Then read S and σ at that point, referring to the two families of curves. Interpolate if necessary. S and σ are the magnitude and angle of sin z.

For example, sin $(1 - j1)$ is not greatly different from $1.5 \underline{/-30°}$, for the intersection of the lines $x = 1$ and $y = -1$ is close to the intersection of the lines $S = 1.5$ and $\sigma = -30$ degrees. Interpolation will give a more accurate reading. (The correct value is $1.45 \underline{/-26.1°}$.) For comparison, this same value of sin $(1 - j1)$, or of sin $1.41 \underline{/-45°}$, can be found in Fig. 4-3. It is evident that Fig. 4-4 gives a neater presentation of the data.[6]

It is remarkable that the chart of Fig. 4-4, plotted for the sine function, can also be used for the cosine. This follows from the fact that $\sin z = \cos (z - \pi/2)$. It is only necessary to adjust the horizontal scale of the chart by subtracting $\pi/2$ to have a chart of the cosine. In Fig. 4-4, the scale to be used in finding the cosine is given at the bottom of the chart, and the scale that applies in finding the sine is given at the top, and they differ by $\pi/2$. Vertical coordinate lines are drawn for integer values of x for finding the sine, and not for x' for finding the cosine; this is merely to avoid a confused appearance of the chart and actually the chart is equally applicable to either function.[7]

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[6] Some of the characteristics of Fig. 4-4 are as follows. Values of sin z read along the horizontal axis ($y = 0$) give the ordinary circular sine of a real variable (sin x). Values of magnitude read along the vertical axis ($x = 0$) give the hyperbolic sine.

Near the origin, where z is small, sin z is approximately equal to z. This is more clearly evident if the reader imagines, or actually sketches in the figure, concentric circles and radial lines to show magnitude and angle of z as expressed in the polar form.

sin z is periodic; that is, $\sin z = \sin (z + 2\pi)$. Also it is odd, for $\sin z = -\sin(-z)$. These relations make it possible to extend the chart indefinitely to the right or left; or, what is the same thing, they make it possible to use the section of chart that is given to find sin $(x + jy)$ for any value of x. Indeed, with the aid of these relations and the fact (from footnote 4) that $\sin \bar{z} = \overline{\sin z}$, a strip of chart from $x = 0$ to $x = \pi/2$ and for only positive y can be used to evaluate sin $(x + jy)$ for any value of x and either positive or negative y. The absolute value of y, of course, must not fall beyond the vertical range of the chart.

[7] To prove for a complex function that $\sin z = \cos (z - \pi/2)$,

$$\begin{aligned} \sin z = \sin (x + jy) &= \sin x \cosh y + j \cos x \sinh y \\ &= \cos\left(x - \frac{\pi}{2}\right) \cosh y - j \sin\left(x - \frac{\pi}{2}\right) \sinh y \\ &= \cos\left(x - \frac{\pi}{2} + jy\right) = \cos\left(z - \frac{\pi}{2}\right) \qquad (4\text{-}44) \end{aligned}$$

An outstanding feature of this chart, and perhaps its greatest advantage, is that there is no uncertainty regarding sign. If x and y are known, S and σ are indicated without ambiguity.

Another method of plotting the same data is shown in Fig. 4-5. Here x and y are read from the curvilinear coordinate system, and the real and imaginary components of sin $(x + jy)$ are read from the rectangular system. The following notation is convenient:

$$\sin (x + jy) = u + jv \tag{4-46}$$

If

$$x + jy = z \tag{4-47}$$

and

$$u + jv = w \tag{4-48}$$

then

$$\sin z = w \tag{4-49}$$

It will be seen that Fig. 4-4 is a plot of w from Eq. 4-49 in the complex z plane, and Fig. 4-5 is a plot of z in the complex w plane. Only one quadrant of the full chart is shown in the figure,[8] as the other three quadrants are symmetrical with it.

Hence if $x - \pi/2 = x'$,

$$\sin (x + jy) = \cos (x' + jy) \tag{4-45}$$

The lower scale in Fig. 4-4 is x', and the origin of the cosine chart is displaced by $\pi/2$ to the right of the origin of the sine chart.

Some of the characteristics of the cosine chart are as follows. Values of cos z read along the horizontal axis give the ordinary circular cosine of a real variable. Values read along the vertical axis give the hyperbolic cosine. Near the origin, where x' and y are small, the cosine is approximately $1\underline{/0°}$. The cosine function is periodic like the sine function. It is an even function, however, and $\cos z = \cos (-z)$.

It is interesting to imagine that the lines of constant S in Fig. 4-4 are the contour lines of equal elevation of a surface above the plane of the figure. It is instructive to visualize such a surface, and even to draw a perspective sketch of it. (Such a drawing is given by Jahnke and Emde, "Tables of Functions with Formulae and Curves," p. 67 of Addenda, Ref. 11). Lines of constant σ, being an orthogonal system, are gradient lines in the three-dimensional picture. Referring to the sine coordinates, there are conical sinks at the origin, at π, 2π, and so on. There is a saddle surface at $\pi/2$ and at other points midway between sinks. The surface rises steeply (exponentially, in fact) at the top and bottom of the chart.

[8] This quadrant of the w plane is equivalent to the single panel of the z plane described in footnote 6.

Lines of equal x are drawn for decimal fractions of π rather than for integer values of x; this makes it easier to use the chart for values of x greater than $\pi/2$. Lines of equal x are hyperbolas, lines of equal y are ellipses.[9] Signs to be attached to x, y, u, and v are not determined from the chart, but from the accompanying table.

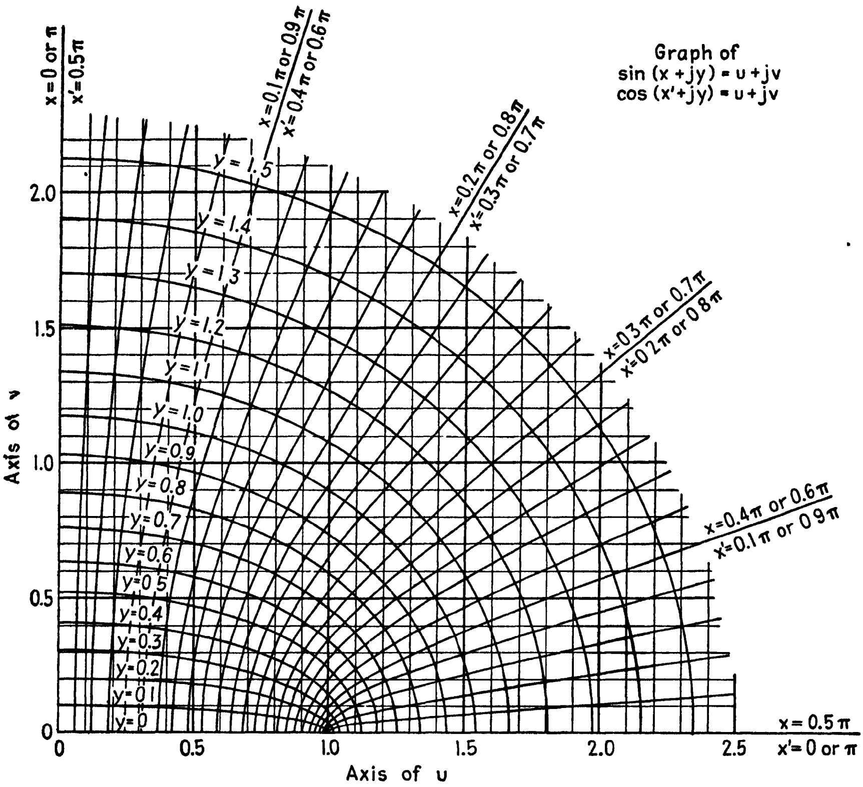

FIG. 4-5. Chart of complex sine and cosine.

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[9] *Proof*: By Eq. 4-26

$$u = \sin x \cosh y \tag{4-50}$$

$$v = \cos x \sinh y \tag{4-51}$$

$$\frac{u^2}{\sin^2 x} = \cosh^2 y \qquad \frac{u^2}{\cosh^2 y} = \sin^2 x \tag{4-52}$$

$$\frac{v^2}{\cos^2 x} = \sinh^2 y \qquad \frac{v^2}{\sinh^2 y} = \cos^2 x \tag{4-53}$$

TABLE 4-1
AUXILIARY DATA FOR FIG. 4-5

$$\sin(x + jy) = u + jv \qquad \sinh(y + jx) = v + ju$$

$$\cos(x' + jy) = u + jv \qquad \cosh(y - jx') = u + jv$$

$$x' = x - \frac{\pi}{2}$$

For *sin* or *sinh*, when x is in the range	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	π to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π	2π to $\frac{5\pi}{2}$
For *cos* or *cosh*, when x' is in the range	$-\frac{\pi}{2}$ to 0	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	π to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π
If y is + →	u is + v is +	u is + v is −	u is − v is −	u is − v is +	u is + v is +
If y is − →	u is + v is −	u is + v is +	u is − v is +	u is − v is −	u is + v is −

To find sin z, the components x and y must be known. Divide x by π to get a value with which to enter the chart. (If x/π is greater than 0.5 but less than 1.0, consider that the scale is folded back as indicated. If the value is greater than 1, subtract any integer and enter the chart with the remainder.) On the two families of curves, find the point corresponding to x and y. At this point read u and v from the rectangular coordinates; these are the real and imaginary components of sin z.

To find cos z, the real part of z is called x', and the scale for x' on the hyperbolic curves is used instead of the scale for x. The real and imaginary components, x' and y, are used to find the cosine in exactly the same way that x and y are used to find the sine.

The auxiliary table for determining signs requires little explanation. As an example of its use, consider sin $(2 - j1)$; here x is between $\pi/2$ and π, and y is negative; hence u is positive and v is positive. This may be

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \tag{4-54}$$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \tag{4-55}$$

Equation 4-54 with constant x gives hyperbolas in the w plane. Equation 4-55 with constant y gives ellipses in the w plane.

verified by comparison with Fig. 4-4. Indeed, the auxiliary table is essentially a skeleton of Fig. 4-4, showing no curves but only giving the sign of the function in each panel of the chart.

Among the many possible ways of plotting the sine and cosine of complex functions, only three of which are shown here, the chart of Fig. 4-5 seems the most generally useful for computation. This is partly because it is simple and symmetrical, and partly because it is available in large, accurate charts.[10] For many ordinary uses the chart of Fig. 4-5 gives sufficient accuracy if larger charts are not available. For accuracy beyond the scope of charts, computation from Eqs. 4-25 and 4-26 is recommended.

Charts of *complex hyperbolic functions* are similar to charts of circular functions; indeed, the similarity is so great that the same charts can be used if interpreted correctly. Consider Fig. 4-5. This is a chart relating x and y to u and v where, by Eq. 4-26,

$$u = \sin x \cosh y \tag{4-56}$$

$$v = \cos x \sinh y \tag{4-57}$$

However, by Eq. 4-35,

$$\begin{aligned}\sinh (y + jx) &= \sinh y \cos x + j \cosh y \sin x \\ &= v + ju \end{aligned} \tag{4-58}$$

Hence this same chart can be used to find the hyperbolic sine: call the real part of the argument y and the imaginary component x; enter the chart with these values and find v and u, and these are the real and imaginary components, respectively, of the hyperbolic sine.

The chart of Fig. 4-5 can also be used to find the hyperbolic cosine, using the scale for x' on the hyperbolas. Since $x' = x - \pi/2$, Eqs. 4-56 and 4-57 can be written

$$u = \cos x' \cosh y \tag{4-59}$$

$$v = -\sin x' \sinh y \tag{4-60}$$

Then, from Eq. 4-36,

$$\begin{aligned}\cosh (y - jx') &= \cosh y \cos x' - j \sinh y \sin x' \\ &= u + jv \end{aligned} \tag{4-61}$$

This means that to find the hyperbolic cosine of a quantity, the real component of the quantity is called y and the negative of the imaginary component is called x'. The chart is entered with these values of y and x', and u and v, the real and imaginary components of the desired hyperbolic cosine are read.

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[10] "Chart Atlas of Complex Hyperbolic and Circular Functions" by A. E. Kennelly (Ref. 12) is an excellent presentation of charts of this kind and of other useful forms also. Although Kennelly's charts are primarily arranged for hyperbolic functions, it is easy to use them for circular functions also. The same author has published "Tables of Complex Hyperbolic and Circular Functions" (Ref. 13).

PROBLEMS

4-1. Derive Eq. 4-12 from the exponential form.

4-2. Compute $\cos z$ for the following values of z: $0.5\underline{/-25^\circ}$; $1.0\underline{/-25^\circ}$; $1.5\underline{/-25^\circ}$. For each value of z, use the following methods:

a. By infinite series (Eq. 4-15) using four terms.

b. By expansion in exponential form (Eq. 4-16).

c. By expansion in trigonometric form (Eq. 4-25).

Locate the results on the charts of Fig. 4-2*a* and *b*.

4-3. Compute $\sin z$ for the values of z given in Prob. 4-2, using for each the three methods specified in that problem. Locate the results on the charts of Fig. 4-3*a* and *b*.

4-4. Compute $\cosh z$ and $\sinh z$ for $z = 1.0\underline{/65^\circ}$, using any one of the three methods specified in Prob. 4-2.

4-5. Compute $\cosh z$ and $\sinh z$ for $z = 0.5\underline{/65^\circ}$, using any one of the three methods specified in Prob. 4-2.

4-6. Starting from Formulas 1 to 7 of Table II, show that Formulas 8, 9, 12, 14, 15, and 16 are correct.

4-7. Starting from Formulas 1 to 7 of Table II, derive Formula 25.

4-8. Using Formulas 1 to 7 of Table II, show that $\sin \bar{z} = \overline{\sin z}$ and $\cos \bar{z} = \overline{\cos z}$ (see footnote 4).

4-9. Read $\sin z$ from the charts of Fig. 4-3 for $z = 1 - j0.5$ and $z = 2 - j1.0$. Repeat using Fig. 4-4, and again using Fig. 4-5. Compute $\sin z$ (by any desired method) for comparison.

4-10. Read $\cos z$ from charts, as in Prob. 4-9. (It may be helpful to draw appropriate vertical cross-sectional lines in Fig. 4-4). Compute $\cos z$ for comparison

4-11. Read $\sin z$ and $\cos z$ from Fig. 4-5 for the following values:

$$z = 1.0 + j0 \qquad z = 0.8 + j0.6 \qquad z = -1.5 - j0.1$$

$$z = 0 + j1.0 \qquad z = 0.5 - j1.2 \qquad z = 0.1 + j1.5$$

4-12. Read $\sinh z$ and $\cosh z$ from Fig. 4-5 for the values of z given in Prob. 4-11.

4-13. Find z from Fig. 4-5, given that

$$\cos z = 1 + j1 \qquad \sin z = 0.7 - j0.6$$

$$\cos z = -1.2 + j0.5 \qquad \sin z = 1.0 - j0.1$$

(Any of the possible values of z will be acceptable.)

4-14. Given $z = 0.4 + j0.3$, find $\tan z$ by using Formula 25*a* of Table II. Check the result by using Formula 50 of Table II.

4-15. Given $z = 1.2 + j0.15$, find $\tan z$ by using Formula 25*a* of Table II. Check the result by using Formula 50 of Table II.

CHAPTER 5

LINE PARAMETERS

1. Range of Values of Parameters. The previous chapters have discussed the derivation of the differential equations of a transmission line and the solution of the differential equations expressed in exponential functions, in trigonometric functions, and in hyperbolic functions.

Certain quantities appear as parameters in these equations. The parameters that have been used most frequently are Z_0, γ, B, α, and β. These parameters are combinations of the so-called line constants l, c, r, and g; the parameters are used for convenience and brevity of notation, but they are more fundamental than mere abbreviations and have physical meaning in connection with transmission-line behavior. This chapter will relate primarily to the line parameters, and the next two to the constants from which the parameters are derived.

First, it is helpful to consider the practical range of values of the line parameters, and it is possible to reach certain fairly general conclusions, at least for specific types of lines.

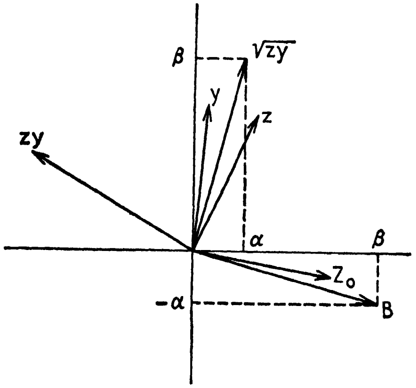

FIG. 5-1. Angular relations on open-wire telephone line or power line.

The constants l, c, r, and g are always positive. This implies that z, the impedance per unit length of line, is a complex quantity in the first quadrant of the complex plane. The admittance y is likewise a complex quantity in the first quadrant.

At radio frequency, the reactance, $2\pi fl$, is usually so much larger than r that the impedance z is very nearly imaginary. Similarly, $2\pi fc$ is much greater than g, and the admittance y is practically imaginary.

At voice frequency, in the neighborhood of 1,000 cycles, the reactance of an open-wire telephone line is several times the resistance; z has an angle in the complex plane of 70 or 80 degrees. Since loss from line to line is quite small, the angle of y is nearly 90 degrees. Representative values of z and y are shown in Fig. 5-1.

At power frequency the reactance of a transmission line is rather low, but the resistance is even lower. Lines that are long enough to require treatment as circuits with distributed parameters are designed to carry large amounts of power, as long lines are usually economic only with heavy loads. Consequently long power lines have rather large conductors, and the resistance is correspondingly low. The reactance of a long, high-voltage power line is typically five to eight times its resistance. Figure 5-1 is therefore reasonably applicable to a high-voltage power line as well as to an open-wire telephone line.

Figure 5-1 can also be used in considering angular relations on radio-frequency lines, although for the typical high-frequency line it greatly exaggerates the effect of loss. That is, y and z would typically be at angles hardly to be distinguished from 90 degrees.

When the angles of y and z are as shown in Fig. 5-1, the product zy has an angle approaching 180 degrees, as indicated, and its square root is at an angle of something less than 90 degrees.[1] Since $\sqrt{zy} = \gamma = \alpha + j\beta$, α and β are the components of $\sqrt{zy}$ as shown in the figure. Note that for typical lines, α is smaller than β.

The quantity B is $-j\sqrt{zy}$ (by Eq. 4-11). This implies that B has the same magnitude as $\sqrt{zy}$ but is displaced from it by -90 degrees as shown in the figure. The components of B, from $B = \beta - j\alpha$, are also shown in the figure. On ordinary lines, B is nearly a real quantity, approximating β. B, however, has a small negative angle unless the line is ideally without loss, in which case B equals β.

The magnitude of B is mainly dependent on the frequency. Both z and y are approximately proportional to frequency and so, therefore, is B. The phase constant β is even more nearly proportional to frequency. This is interesting in its relation to the velocity of propagation of waves on the line. It was shown in Eq. 1-49 that

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[1] The negative of the vector shown in Fig. 5-1 as $\sqrt{zy}$ is, of course, also a square root of zy. In deciding which of the alternative values of the square root to use for γ, we must be consistent throughout the mathematical development of the subject. If it were decided to select the value of $\sqrt{zy}$ that lies in the third quadrant in Fig. 5-1, α and β would both be negative. This would require a different physical interpretation, and much of the discussion of Chaps. 1 and 2 would have to be altered, for it was there tacitly accepted that α and β are positive. The mathematics of transmission lines could be developed on the opposite assumption, but there is no advantage, it is not customary, and the complications are greater. Therefore we shall now explicitly decide to use for γ the value of $\sqrt{zy}$ that falls in the first quadrant.

In this connection, note that $jB = \gamma = \sqrt{zy}$, and hence $B = \sqrt{zy}/j = -j\sqrt{zy}$. It is mathematically correct to write $B = \sqrt{-zy}$, but this form should be carefully avoided because it suggests that B could be written $j\gamma$, instead of $-j\gamma$. To write $B = j\gamma$ is incorrect because it is inconsistent with the assumptions made above.

$$\text{Velocity} = \frac{2\pi f}{\beta} \tag{5-1}$$

Hence, in the range in which β is approximately proportional to frequency, velocity is nearly independent of frequency.

Indeed, it was shown in Sec. 1, Chap. 3, that the velocity of waves on a transmission line is equal to the speed of light if waves travel on the line without loss and if the term containing K_φ in the inductance formulas of Table I can be neglected. This latter condition is met if there is negligible magnetic field within the line conductors, owing either to high frequency and hence marked skin effect or to a large ratio of line spacing to conductor radius.

Conditions that make the wave velocity approach the speed of light are approximated most closely at high frequency. At low frequency, loss is relatively more important and there is more magnetic field within the conductor; both of these effects tend to slow the wave. It is not surprising, therefore, that velocity is low at low frequency, increases to some extent with increasing frequency, and then approaches the speed of light as frequency is indefinitely increased. Velocity of wave propagation on a typical open-wire telephone line is shown in Fig. 5-2. It is plotted as a function of frequency, and is seen to be nearly constant above 1,000 cycles.

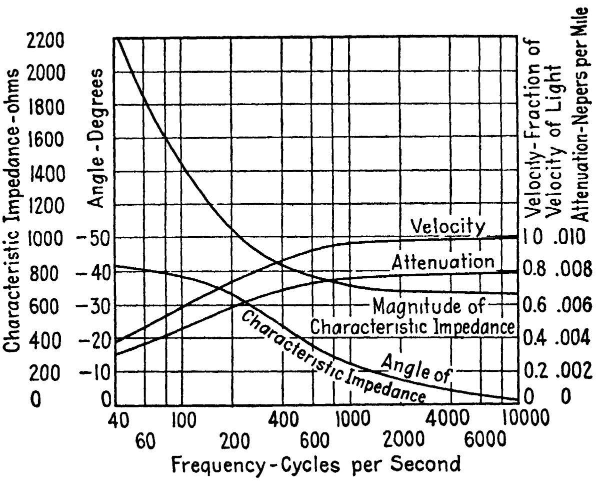

FIG. 5-2. Phase velocity, attenuation, and characteristic impedance of a typical telephone line.

Open-wire line	r = 10.44 ohms per mile
12-inch spacing	l = 3.66 millihenrys per mile
104-mil copper	c = 0.00838 microfarad per mile
Neglecting skin effect	$g = 0$

Velocity on a high-voltage power line (the Hoover Dam line) is shown in Fig. 5-3. Resistance is low because of the large cross section of the copper conductors. Because the conductors are tubular, they have no internal magnetic field. Hence, the velocity of propagation is nearly the speed of light for all frequencies above 50 cycles.

Attenuation depends on α, which is the imaginary component of B, or the real component of γ. Large loss gives rapid attenuation, and α is greatly larger for the telephone line of Fig. 5-2 than for the power line of Fig. 5-3. Generally speaking, attenuation increases slightly with increasing frequency in the lower frequency range and becomes practically constant in the higher frequency range. (It will be shown in Chap. 12 that α at high frequency increases as the square root of frequency, but at the highest frequencies of Figs. 5-2 and 5-3 this slow increase, the result of skin effect, has not begun to appear.)

Dimensionally, α is measured in inverse length; αx is a dimensionless number, as indeed it must be to be used as an exponent or as the argument of a trigonometric function. For convenience the term *nepers* is frequently used in speaking of αx. The value of α is given as so-many nepers per mile or nepers per meter, depending on whether x is measured in miles or meters. The neper, however, like radians or degrees or decibels, is

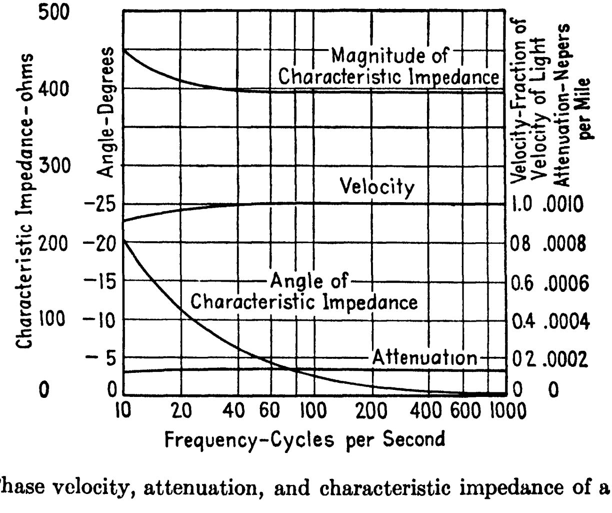

FIG. 5-3. Phase velocity, attenuation, and characteristic impedance of a high-voltage power line.

Three-phase line
390-inch spacing in plane
512,000 c.m. copper cross section, segmental hollow conductor
Outside diameter: 1.40 inches

r = 0.113 ohms per mile, each wire
l = 0.00212 henry per mile, each wire
c = 0.0136 microfarad per mile, each wire
$g = 0$
Neglecting skin effect but considering thin walls of conductor

dimensionless; it merely gives a name to a ratio of dimensionally similar quantities.[2]

Returning to Fig. 5-1, the characteristic impedance of the line, Z_0, can be shown in relation to the other line parameters. Since $Z_0 = \sqrt{z/y}$, and since the angle of z is commonly less than the angle of y although both are typically near 90 degrees, the angle of Z_0 is usually small and negative. It is a useful check on computation to note that the angle of Z_0 must be a somewhat smaller negative angle than the angle of B unless the line admittance is purely imaginary ($g = 0$) in which case they are the same. A typical value of the angle of Z_0, or of B, for either a 60-cycle high-voltage power line or an open-wire telephone line at 1,000 cycles, is about -5 or -10 degrees. The angle is greater at lower frequency, as appears in Figs. 5-2 and 5-3.

The magnitude of the characteristic impedance is more or less independent of frequency, except that it typically increases at low frequency, but unlike the velocity or the propagation constant it is greatly dependent upon the physical dimensions of the line. If the spacing between wires is large, or if the wires are small, z is great and y is small; Z_0 is consequently large. If, on the contrary, the conductors are large and close together, z is small, y is large, and the characteristic impedance is quite low.

As an example, the characteristic impedance of the telephone line of Fig. 5-2 is about 700 ohms at 1,000 cycles. The characteristic impedance of the power line of Fig. 5-3 is about 400 ohms, but this is Z_0 *to neutral*; the characteristic impedance of a two-wire line with the same equivalent spacing and the same conductors would be twice as much, or about 800 ohms.[3] The ratio of b/a for the power line is roughly twice that for the telephone line, and the comparable characteristic impedances are 800 and 700 ohms, respectively, for two-wire lines.

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[2] It will be seen in Chap. 9 that nepers and decibels are both capable of interpretation as ratios of power input to power output if applied to a single pure traveling wave. Attenuation in nepers can be changed to attenuation in decibels by multiplying by 8.686. This factor can be used to change the attenuation scale in Fig. 5-2 or 5-3 to decibels per mile *if* the line is terminated in its characteristic impedance and there is no reflected wave.

[3] When inductance and capacitance to neutral are used in computing Z_0, the result is a characteristic impedance *to neutral*. This may be visualized by considering a three-phase line terminated in its characteristic impedance; such a line has a Y-connected three-phase load, the impedance of each arm of the Y being the characteristic impedance of the line to neutral.

If a two-wire line is terminated in its characteristic impedance, it may either be loaded with a single impedance equal to the characteristic impedance of the two-wire line, or it may be loaded with two impedances in series, each of which is equal to the characteristic impedance of the line to neutral. The mid-point, between the two impedances, is visualized as being the neutral point (see also Sec. 10, Chap. 6).

It is usual for the characteristic impedance of a high-voltage power line to be in the neighborhood of 400 ohms to neutral. For an open-wire telephone line, Z_0 is commonly 600 to 700 ohms for the two-wire line.

Coaxial lines for radio-frequency transmission usually have much lower impedance. It will be shown in Chap. 12 that a line with a characteristic impedance of about 70 ohms is particularly advantageous because of low loss. By having very small spacing between the inner and outer conductors, it is possible to design a coaxial line with characteristic impedance as low as a fraction of an ohm.

2. Typical Cable Parameters. Frequently the pair of wires that constitutes a telephone transmission line is grouped with many other similar pairs and enclosed within a protecting sheath, as of lead, to form a cable. A cable may contain only a few pairs of wires, or hundreds or even thousands of pairs. A telephone cable may be suspended from poles, but it may also be run in ducts underground. The outstanding advantages of the use of cable are the saving of space that results, and protection of the circuits from the weather. Also, in cities, underground construction is usually required by law.

Power circuits are frequently run underground in cable, also, and for much the same reasons. Sometimes one conductor and sometimes three conductors are enclosed within a protective sheath, properly insulated from the sheath and from each other.

Cables, whether for power or telephone work, have close spacing of the conductors. This results in relatively low inductance and high capacitance. Conductors in cables are made small to save expense; therefore resistance is relatively high. Insulation is a comparatively thin layer of solid material, or of solid material impregnated with oil or some other insulating liquid, or with gas under pressure, and as a result the line-to-line conductance and often the dielectric losses are appreciable.

Because of the close spacing, characteristic impedance of a cable is low. Because of high loss, attenuation is large. Because the insulating material has a dielectric constant greater than that of air, the velocity of propagation is less than on open-wire lines.

The behavior of power cable as a long transmission line is not often of interest, for power cables are always very short compared to the wavelength of power frequencies. In telephone practice, however, long cables are becoming continually more important, both for transmission of voice frequency and for carrier-current operation.

Comparing telephone cable with open-wire telephone line, the resistance is of the order of magnitude of eight times as much, inductance (of *unloaded* cable) is about one-fourth, capacitance about eight times, and conductance is five times or more. Because of the high resistance and low inductance, the impedance per unit length, z, of a cable is a quantity more nearly real than imaginary. The admittance, however, remains almost imaginary. This means that zy for a cable has an angle of only slightly more than 90 degrees, as indicated in Fig. 5-4, and $\sqrt{zy}$ is at about 45 degrees. B is at about -45 degrees, and α is nearly as great as β. These comparisons are valid in the voice-frequency range, with 1,000 cycles as typical.

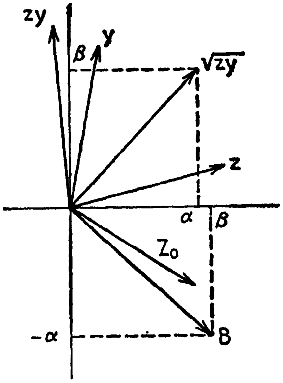

FIG. 5-4. Angular relations, telephone cable.

As shown in Fig. 5-5, the characteristic impedance, wave velocity, and attenuation of a cable are all quite dependent upon frequency. This is mainly because the z of the cable is largely resistance throughout the voice-frequency range and hence is by no means proportional to frequency, whereas y varies approximately with frequency.

Because of the variation with frequency it is hard to compare cable parameters with those of an open-wire line, but if the voice-frequency range for ordinary telephone transmission of speech is taken to be 200 to 2,700 cycles, some idea of relative values can be suggested. Very roughly, cable attenuation is ten or more times that of open-wire telephone line; velocity on the cable is less than one-third the speed of light; and char-

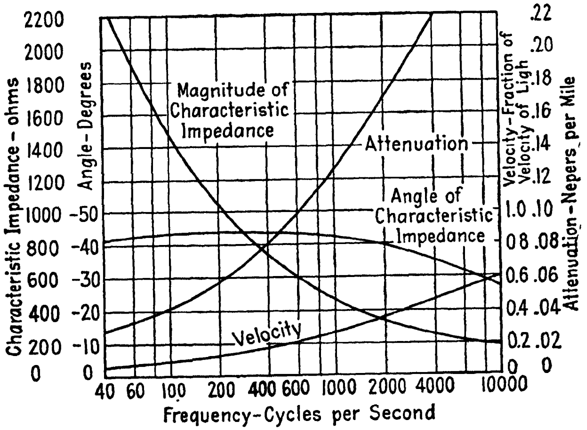

FIG. 5-5. Phase velocity, attenuation, and characteristic impedance of a typical telephone cable.

Nonloaded
Paper insulated
No. 19 AWG conductors

r = 83.4 ohms per mile, both wires
l = 0.0010 henry per mile, both wires
c = 0.062 microfarad per mile, both wires
g = 1.5 micromhos per mile, both wires

acteristic impedance of a cable centers around 400 or 500 ohms in magnitude, with an angle of 40 to 45 degrees.

3. Computation of Parameters. Computation of Z_0, the characteristic impedance, for any transmission line, offers no particular problems. It is $\sqrt{z/y}$, and can be computed from

$$Z_0 = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \tag{5-2}$$

At high frequency, with large ω, the characteristic impedance approaches the purely real value of

$$Z_0 = \sqrt{\frac{l}{c}} \tag{5-3}$$

Since, at high frequency, skin effect is quite fully developed, the values of l and c from formulas of Table I can be substituted into Eq. 5-3 to give:

For parallel-wire line,

$$Z_0 = \sqrt{\frac{0.921 \log (b/a) \times 10^{-6}}{12.08 \times 10^{-12}/\log (b/a)}} = 276 \log \frac{b}{a} \tag{5-3a}$$

For coaxial line,

$$Z_0 = \sqrt{\frac{0.461 \log (b/a) \times 10^{-6}}{24.16k \times 10^{-12}/\log (b/a)}} = 138 \log \frac{b}{a} \tag{5-3b}$$

These equations neglect any effect of line loss. They are almost exact at radio frequency.

On the other hand, if frequency is extremely low, so that r is much greater than ωl, and g is much greater than ωc, the characteristic impedance approaches

$$Z_0 = \sqrt{\frac{r}{g}} \tag{5-4}$$

This limiting value is also a purely real quantity. It is the limit of the characteristic impedance at frequencies so low that zero-frequency or d-c operation is approached.

Computation of the parameters α and β is possible in several ways. Since they are the magnitudes of the real and imaginary components of $\sqrt{zy}$, it is possible to compute $\sqrt{zy}$ and find its real and imaginary components:

$$\alpha + j\beta = \sqrt{zy} = \sqrt{(r + j\omega l)(g + j\omega c)} \tag{5-5}$$

This is very often the simplest means of computation. However, explicit formulas for α and β are sometimes desired, and they can be derived[4] from Eq. 5-5 as

$$\alpha = \sqrt{\tfrac{1}{2}\left[\sqrt{(r^2 + \omega^2 l^2)(g^2 + \omega^2 c^2)} + rg - \omega^2 lc\right]} \tag{5-6}$$

and

$$\beta = \sqrt{\tfrac{1}{2}\left[\sqrt{(r^2 + \omega^2 l^2)(g^2 + \omega^2 c^2)} - rg + \omega^2 lc\right]} \tag{5-7}$$

In computing a numerical value for α, it is hard to get good accuracy from either Eq. 5-5 or 5-6 if α is much smaller than β. This difficulty arises at high frequency, and in this range it can fortunately be avoided by using a different form of expression for α. Eq. 5-5 is expanded by the binomial formula to give

$$\begin{aligned} \alpha + j\beta &= \sqrt{(r + j\omega l)(g + j\omega c)} \\ &= [(rg - \omega^2 lc) + j\omega(lg + rc)]^{\frac{1}{2}} \\ &= (rg - \omega^2 lc)^{\frac{1}{2}} + \frac{1}{2}\frac{j\omega(lg + rc)}{(rg - \omega^2 lc)^{\frac{1}{2}}} - \frac{1}{8}\frac{[j\omega(lg + rc)]^2}{(rg - \omega^2 lc)^{\frac{3}{2}}} + \cdots \end{aligned} \tag{5-14}$$

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[4] The derivation is as follows. From Eq. 5-5

$$\begin{aligned} (\alpha + j\beta)^2 &= \alpha^2 - \beta^2 + j(2\alpha\beta) \\ &= rg - \omega^2 lc + j(\omega lg + \omega rc) \end{aligned} \tag{5-8}$$

Equating the real and the imaginary components, and introducing the abbreviations M and N for convenience,

$$\alpha^2 - \beta^2 = rg - \omega^2 lc = M \tag{5-9}$$

$$2\alpha\beta = \omega lg + \omega cr = N \tag{5-10}$$

Simultaneous solution gives

$$\alpha^2 - \frac{N^2}{4\alpha^2} = M \qquad \text{or} \qquad \alpha^4 - M\alpha^2 - \frac{1}{4}N^2 = 0 \tag{5-11}$$

$$\alpha^2 = \frac{M \pm \sqrt{M^2 + N^2}}{2} \tag{5-12}$$

whence

$$\alpha = \sqrt{\tfrac{1}{2}\left[(rg - \omega^2 lc) \pm \sqrt{(rg - \omega^2 lc)^2 + (\omega lg + \omega cr)^2}\right]} \tag{5-13}$$

Consider the ambiguity of the ± sign: since the quantity under the inner radical must always be numerically greater than $(rg - \omega^2 lc)^2$, the negative sign would lead to an imaginary value for α. Since, according to our original stipulation in Eq. 1-41, α and β are both real, the negative sign is spurious and only the positive sign is retained. The ambiguity of sign of the outer radical is discussed in footnote 1. Simplification of Eq. 5-13 then leads to Eq. 5-6.

Similarly, eliminating α between Eqs. 5-9 and 5-10, β is found to be as in Eq. 5-7.

If ω is sufficiently large, rg is very small compared to $\omega^2 lc$, and as a good approximation *at high frequency*:

$$\alpha + j\beta = j\omega\sqrt{lc} + \frac{lg + rc}{2\sqrt{lc}} + j\,\frac{(lg + rc)^2}{8\omega(lc)^{\frac{3}{2}}} + \cdots \tag{5-15}$$

Equating the real terms on each side of this equation, the second term on the right-hand side is an approximation of α. There are other real terms in the expansion, but other terms have ω to at least the second power in the denominator. If ω is of an order of magnitude of millions or greater (and possibly if it is of the order of thousands), the term shown is the only one that need be considered. Thus, as an approximation valid at radio frequencies (and possibly at the higher audio frequencies)

$$\alpha = \frac{lg + rc}{2\sqrt{lc}} \tag{5-16}$$

Similarly, equating imaginary terms, the first and third terms on the right-hand side of Eq. 5-15 contribute to β. Since, however, the first term has ω in the numerator, and the third has ω in the denominator, and all successive terms have higher powers of ω in the denominator, only the first term need be retained at sufficiently high frequency, and as an approximation valid at high frequency

$$\beta = \omega\sqrt{lc} \tag{5-17}$$

This same expression for β was obtained in Eq. 3-5; there is, however, a difference in assumptions. Equation 3-5 was derived for a lossless line at any frequency, whereas Eq. 5-17 is for a high-frequency line even though there are losses.

Contrasting to the values obtained at high frequency, approximations for α and β can be found that are valid only *at extremely low frequency.* If, in Eq. 5-14, rg is very large compared to $\omega^2 lc$, we find, as an approximation that is good as zero frequency is approached,

$$\alpha = \sqrt{rg} \tag{5-18}$$

$$\beta = \frac{\omega(lg + rc)}{2\sqrt{rg}} \tag{5-19}$$

From β an expression for velocity of wave propagation can be found:

$$\text{Velocity} = \frac{\omega}{\beta} = \frac{2\sqrt{rg}}{lg + rc} \tag{5-20}$$

With direct current, corresponding to zero frequency, it is inappropriate to speak of waves or wave propagation. Nevertheless, the wave concept can be used for any a-c frequency, however low, and such a quantity as velocity of propagation has meaning for waves of any frequency. It is therefore reasonable to speak of the value approached by the velocity as frequency is reduced toward zero.

If $r/l = g/c$ (see Eq. 9-32), the velocity given by Eq. 5-20 is $1/\sqrt{lc}$, approximately the velocity of light. If either r or g is zero, velocity approaches zero, as in Figs. 5-2 and 5-3. For all other conditions, the velocity approaches some intermediate value between these extremes.

The value of α in Eq. 5-18 finds practical application in connection with d-c lines. At zero frequency, the propagation constant γ is equal to α, for β vanishes.

4. Direct-current Transmission. Voltage and current distribution along a d-c transmission line are found by using these relations for zero frequency in the appropriate transmission-line equations. Equations 4-32 and 4-34 are well adapted to use with a d-c line, and give

$$V = V_r \cosh \sqrt{rg}\, x + I_r \sqrt{\frac{r}{g}} \sinh \sqrt{rg}\, x \tag{5-21}$$

$$I = I_r \cosh \sqrt{rg}\, x + V_r \sqrt{\frac{g}{r}} \sinh \sqrt{rg}\, x \tag{5-22}$$

These equations are needed if a d-c transmission line has both resistance r and conductance g. They describe voltage and current in terms of real hyperbolic functions. If either r or g is zero, it is unnecessary to consider distributed constants, for the one remaining constant can be considered to be lumped.

It is interesting to notice that transmission-line equations in terms of hyperbolic functions are particularly suited to d-c lines, as the equations in terms of ordinary circular trigonometric functions are specially fitted to high-frequency lines.

5. Effect of Parameters on Voltage and Current Distribution. Let us next consider the effect of the various parameters on the behavior of transmission lines. The discussion will show why the combinations of constants that are called Z_0, α, and β are specially significant as parameters.

Characteristic Impedance. The characteristic impedance Z_0 appears in most equations in relation to the load impedance Z_r. The ratio Z_r/Z_0 determines the reflection factor, and therefore the standing-wave ratio of the line. It determines whether resonant peaks of voltage and current appear on the line.

This is illustrated in Fig. 5-6, which shows voltage and current distribution on a line with a series of different values of Z_r/Z_0. Figure 5-6*a* shows the large standing waves that occur on a line with infinite load impedance. Figure 5-6*b* shows the less extreme standing waves on a line with a load that makes $Z_r/Z_0 = 3$; it is intermediate between the open-circuit condition of Fig. 5-6*a* and the total lack of resonance on the *flat* line for which $Z_r/Z_0 = 1$, as in Fig. 5-6*c*. When the load impedance is less than the characteristic impedance, there is resonance, but the standing waves are inverted; this is shown in Fig. 5-6*d* for which Z_r/Z_0 is $\frac{1}{3}$. Finally, with short circuit, $Z_r/Z_0 = 0$, as in Fig. 5-6*e*, and the standing waves are as prominent as on the open-circuited line.

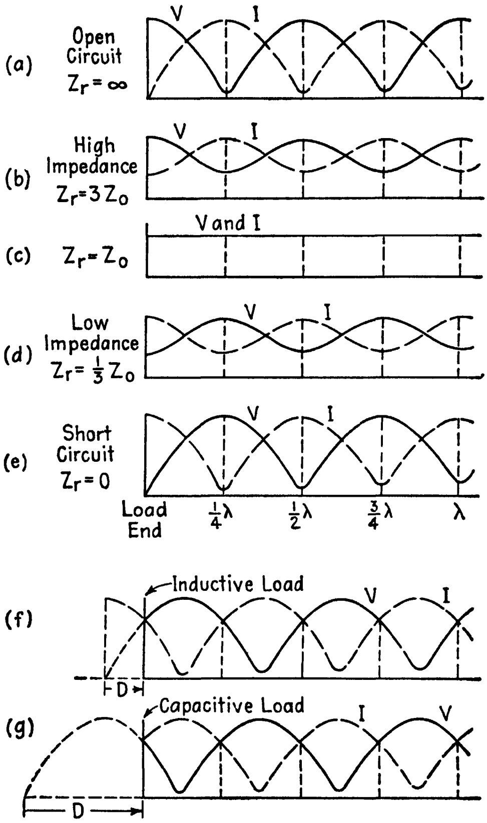

FIG. 5-6. Effect of load on voltage and current distribution.

In Fig. 5-6*a* to *e*, it is assumed that the load impedance has the same angle as has the characteristic impedance of the line. This implies a load that is largely, or entirely, resistive. Figure 5-6*f* and *g* illustrate current and voltage distribution when the load is reactive; these will be discussed later.

Attenuation. Figure 5-7 shows the effect of changing the attenuation from a small value to a large value. Since some specific condition of load on the line must be selected for purposes of illustration, let us assume that the receiving end of the line is open-circuited; this gives maximum standing waves. Curves of voltage distribution are shown, with magnitude of voltage in curves *a* and *b* and angle of voltage relative to angle at the receiver in *c*.

The voltage distribution on such a line is given mathematically by either

$$V = V_r \cos Bx \tag{5-23}$$

or

$$V = V_r \cosh \gamma x \tag{5-24}$$

derived from Eqs. 4-9 and 4-32 by letting $I_r = 0$. These two equations

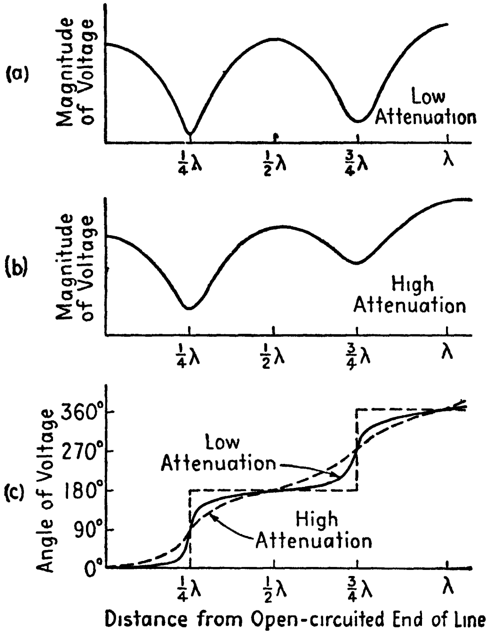

FIG. 5-7. Effect of attenuation on voltage distribution.

express exactly the same information, and either may be used according to personal preference. For an ordinary line, B is nearly a real quantity, whereas γ is nearly imaginary.

The voltage at any point on the line being proportional to the cosine, the voltage distribution curve of Fig. 5-7*a* is merely the curve of the cosine of a complex quantity, Bx. As x increases, Bx increases in magnitude but retains a constant angle in the complex plane. That is, the curve of Fig. 5-7*a* is one of the curves of Fig. 4-2. It is similar to the curve from Fig. 4-2 for $\mu = -5$ degrees.

Voltage distribution along a line with greater loss and higher attenuation is shown in Fig. 5-7*b*. For this curve the angle of B (μ in Fig. 4-2) is perhaps -15 degrees.

Indeed, each of the curves of the chart of Fig. 4-2 can be interpreted as voltage distribution along an open-circuited line, the different curves corresponding to different amounts of loss in the line. The lowest of the family of curves gives voltage distribution with zero loss, and is the circular cosine of a real variable.

The phase angle as well as the magnitude of the voltage at different points along the line can be read from the chart of Fig. 4-2. In Fig. 5-7*c* there are plotted two phase angle curves, one for a line with low attenuation and one for a line with high attenuation.

Current distribution along the open-circuited line is given by Eq. 4-13 or Eq. 4-34 with $I_r = 0$:

$$I = j \frac{V_r}{Z_0} \sin Bx \tag{5-25}$$

or

$$I = \frac{V_r}{Z_0} \sinh \gamma x \tag{5-26}$$

Since Bx is nearly real, $\sin Bx$ is nearly real, and the j in Eq. 5-25 therefore indicates that current is substantially in quadrature with the voltage. (This conclusion is valid except in the neighborhood of quarter-wavelength points.)

Voltage distribution on an open line with low attenuation is repeated in Fig. 5-8. Current distribution is also shown in this figure, being plotted as a curve of $Z_0 I$ (this permits a direct comparison of magnitude between current and voltage). The curve of current distribution, being a curve of the sine of a complex argument, might be taken directly from Fig. 4-3.

The current and voltage curves of Fig. 5-8*a* are contained within envelopes indicated by dash lines. The upper envelope is a real hyperbolic cosine curve and the lower envelope is a real hyperbolic sine curve.[5]

---

[5] Expand either Eq. 5-23 or 5-24 by using the formulas of Table II or by Eq. 4-28 or 4-38, giving

$$V = V_r \cos Bx = V_r \cos (\beta x - j\alpha x) \tag{5-27}$$

or

$$V = V_r \cosh \gamma x = V_r \cosh (\alpha x + j\beta x) \tag{5-28}$$

From either of these,

$$V = V_r (\cos \beta x \cosh \alpha x + j \sin \beta x \sinh \alpha x) \tag{5-29}$$

A helpful way to think of the curves of Fig. 5-8*a* is to consider that the median dash line represents the magnitude of the incident wave $V_1e^{\alpha x}$, as marked. It slopes down from right to left as the incident wave becomes smaller while traveling toward the load. The reflected wave, traveling from left to right, is continually changing in phase relative to the incident

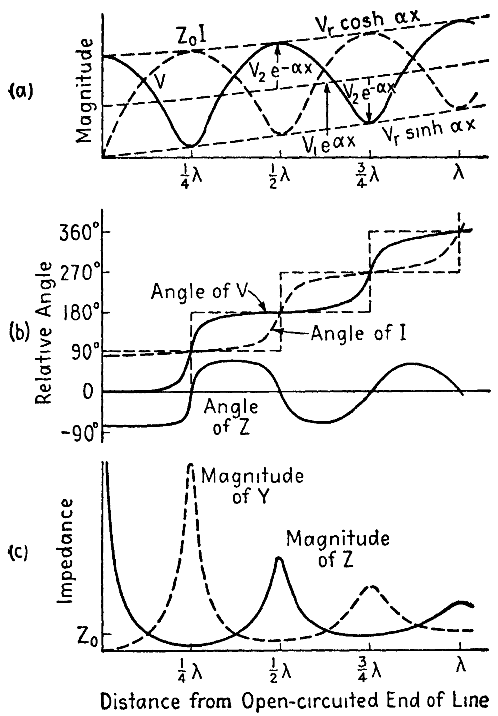

FIG. 5-8. Voltage, current, and impedance distribution with medium attenuation.

The points of tangency to the upper envelope in Fig. 5-8*a* are at the half-wavelength points where $\beta x = 0, \pi, 2\pi$, and so on. At these points $|\cos \beta x| = 1$ and $\sin \beta x = 0$, so, for points on the envelope, Eq. 5-29 gives

$$\text{Upper envelope of } V = V_r \cosh \alpha x \qquad (5\text{-}30)$$

The points of tangency to the lower envelope are at the odd quarter-wavelength points where $\cos \beta x = 0$ and $|\sin \beta x| = 1$. Hence, for points on the lower envelope, Eq. 5-29 gives as the magnitude of V:

$$\text{Lower envelope of } V = V_r \sinh \alpha x \qquad (5\text{-}31)$$

By expanding Eq. 5-25 or 5-26 it is shown that the current distribution curve lies between exactly the same upper and lower envelopes. Mathematically, these curves are envelopes of the voltage and current distribution curves as β varies, α remaining constant.

wave. At the half-wavelength points the incident and reflected waves add; at $\lambda/2$, for example, $V_2e^{-\alpha x}$ adds to the incident wave to give the total voltage, as shown. At odd quarter-wavelength points, on the contrary, the reflected wave is to be subtracted, as indicated at $3\lambda/4$.

From this it will be seen that the median line between envelopes is the magnitude of the incident wave, and the distance between upper and lower envelopes is twice the magnitude of the reflected wave. This concept is quite general; it is not limited to this example of an open-circuited line, but can be applied to any line with any termination. For instance, Fig. 5-9 shows the application of this idea to a long line with moderate attenuation, the reflection factor at the load being something less than 1.

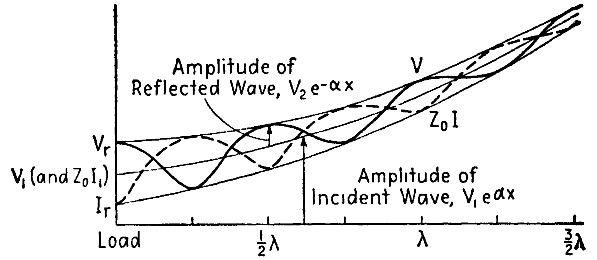

FIG. 5-9. Voltage and current distribution with large attenuation.

Because the reflection factor is less than 1, the magnitude of the reflected wave is everywhere less than the magnitude of the incident wave. At great distances from the load the reflected wave is small, whereas the incident wave is large, so the maxima and minima of resonance are less pronounced.

If the distance from the load is so great that αx is as much as 1 or 2, the reflected wave is quite overshadowed by the incident wave. Conditions at such a distance, and from there on toward the sending end, are practically as if no reflection had been produced at the load; the reflected wave has virtually disappeared in this region, and the curves of voltage and current distribution are hardly to be distinguished from exponential curves.

It is interesting that on any line with loss, if the line is long enough, conditions at a great distance from the load will be similar to those on a line that is terminated in its characteristic impedance. This tendency, seen in Fig. 5-9, applies to angle and impedance, as well as to magnitude of voltage and current. This gives the concept of the *infinite line,* a line so long that behavior near the sending end is quite unaffected by conditions at the receiving end. It is unnecessary to have any information about the receiving end if the line can be assumed to be infinitely long beyond the point at which its action is being studied. No reflections will return, and the part of the line under investigation carries an incident wave only.

6. Effect of Attenuation on Impedance. On a line with loss, the input impedance is somewhat affected by the attentuation. Curves of impedance are similar to those of Figs. 3-1, 3-2, and 3-5, but as a result of attentuation the magnitude of impedance is never zero, and never infinite, and the angle of the impedance does not change abruptly. Moreover, the resonant maxima and minima of impedance become less marked at increased distance from the load.

An example is shown in Fig. 5-8*c*. An open-circuited line is used for this illustration, but the discussion may be extended to a line with any termination.

The impedance of any line with loss is found by dividing V by I, as was done in deriving Eq. 3-36. Dividing Eq. 4-8 by 4-12,

$$\frac{Z}{Z_0} = \frac{\dfrac{Z_r}{Z_0} \cos Bx + j \sin Bx}{\cos Bx + j \dfrac{Z_r}{Z_0} \sin Bx} \tag{5-32}$$

(This is the same as Eq. 3-36, except that loss is taken into account by using B instead of β.) For an expression in hyperbolic functions, divide Eq. 4-31 by 4-33:

$$\frac{Z}{Z_0} = \frac{\dfrac{Z_r}{Z_0} \cosh \gamma x + \sinh \gamma x}{\cosh \gamma x + \dfrac{Z_r}{Z_0} \sinh \gamma x} \tag{5-33}$$

When applied to an *open-circuited line*, as in Fig. 5-8, Eq. 5-32 reduces to

$$\frac{Z}{Z_0} = \frac{\cos Bx}{j \sin Bx} = -j \cot Bx \tag{5-34}$$

or Eq. 5-33 to

$$\frac{Z}{Z_0} = \frac{\cosh \gamma x}{\sinh \gamma x} = \coth \gamma x \tag{5-35}$$

(In Eq. 5-35, it must be remembered that the hyperbolic cotangent of an imaginary quantity is imaginary and negative.)

Impedance at the open-circuited end of the line is, of course, infinite. At ½ wavelength the impedance is again high (see Fig. 5-8*c*). Another maximum occurs at 1 wavelength. The second maximum of impedance is not as high as the first, for the current (as in Fig. 5-8*a*) does not reach as low a minimum at λ as at $\lambda/2$. Indeed, the amplitude of current at the minimum is proportional to sinh αx, and since α is rather small, sinh αx

is approximately equal to αx. Hence the current at a minimum point is approximately proportional to the distance of that point from the load. Since the voltage at λ is of the same order of magnitude as voltage at $\lambda/2$, it follows that impedance at λ is about half the impedance at $\lambda/2$.

There are points of low impedance at $\frac{1}{4}$ wavelength and at $\frac{3}{4}$ wavelength. For reasons analogous to those discussed, the minimum at the latter point is about three times as high as the minimum at the former. Thus it is clear that a line that is open-circuited at the receiving end has lower impedance if it is $\frac{1}{4}$ wavelength long than if it is $\frac{3}{4}$ wavelength long.

The angle of the impedance of the open-circuited line is shown in Fig. 5-8*b*. It is nearly -90 degrees for a short line; the impedance, that is, is almost pure capacitive reactance. In the second quarter-wavelength the angle is positive and the impedance is inductive. Each quarter-wavelength the angle swings back and forth from positive to negative, but each time the swing is less extreme, and the angle of the input impedance to a long line does not reach a value of even approximately 90 degrees.

An important fact that must not be overlooked is that if the impedance is at maximum or at minimum (at the quarter-wavelength points on the line) the impedance is not reactive but resistive. At these points the current and voltage are in phase, as shown in Fig. 5-8*b*. It is immaterial how small the loss in the line; if an open- or short-circuited line is exactly a multiple of a quarter-wavelength long, it has the properties of a high or low *resistance.*

An aspect of the impedance that is occasionally misinterpreted is the appearance of sharpness of the peaks of high impedance, as at $\lambda/2$ in Fig. 5-8*c*, and the seeming lack of sharpness of the minima, as at $\lambda/4$. At first glance it looks as if it would be necessary to adjust the length of a line (or the frequency) quite exactly to get the full advantage of the impedance peak at $\lambda/2$, whereas a line anywhere near $\lambda/4$ in length would be good enough for providing a low impedance. This is not true. Let us plot the admittance, which may be called Y, the reciprocal of the impedance Z. Y is the sending-end admittance of a line. Referring to the curves of Fig. 5-8*a* admittance is the ratio of the current to the voltage, I/V. When derived in this way it is clear that the admittance curve will have maxima approximately as high and as sharp at the odd quarter-wavelength points as the impedance curve has at the even quarter-wavelength points, height and sharpness of both being dependent on distance from the load. The curve for Y is actually a plot of the magnitude of the function[6] tan Bx.

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[6] For numerical computation, the tangent of a complex quantity is most easily obtained from Formula 25*a* of Table II, and the cotangent is the reciprocal of the tangent. Charts of the tangent and cotangent of complex variables can be plotted

Equation 5-34 gives the impedance of an open-circuited line. The impedance of a *short-circuited* line is, correspondingly, from Eq. 5-32,

$$\frac{Z}{Z_0} = j \tan Bx = \tanh \gamma x \tag{5-36}$$

with which Eq. 3-37 may be compared.

A short-circuited line has high input impedance if it is $\frac{1}{4}$ wavelength long. The input impedance of a short-circuited quarter-wavelength line is about twice that of an open-circuited half-wavelength line, assuming the same line and the same frequency. Curves of Fig. 5-8 can be applied to the short-circuited line if voltage and current are interchanged, and also impedance and admittance.

7. Equivalent Short-circuited Line. In general, a transmission line is terminated in a load, the impedance of which we may call Z_r. It is an interesting concept that the behavior of the line will be in no way altered if the receiving end of the line is connected, not to the load of impedance Z_r, but instead to an additional length of line with input impedance equal to Z_r. In previous discussion we have mentioned the possibility of substituting a lumped impedance for a section of transmission line; it is now suggested that a section of transmission line may be substituted for a lumped impedance.

Let us consider the mathematics that leads to such an interpretation. The numerator and the denominator of Eq. 5-32 are each of the form of Formula 28 of Table II. By transforming the numerator of Eq. 5-32 into a sine form and the denominator into a cosine form, and dividing, we obtain the following expression for impedance:

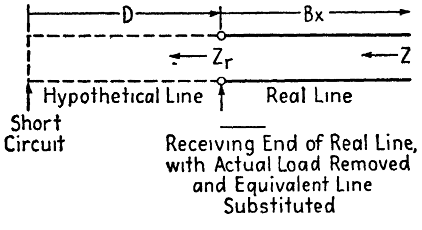
D
Bx
Zr
Z
Hypothetical Line
Real Line
Short Circuit
Receiving End of Real Line, with Actual Load Removed and Equivalent Line Substituted

FIG. 5-10. Substitution of a hypothetical short-circuited line for a lumped load.

$$\frac{Z}{Z_0} = j \tan (Bx + D) \tag{5-37}$$

where D is defined by

$$\tan D = \frac{Z_r}{jZ_0} \tag{5-38}$$

Equation 5-37 has the appearance of an expression for the impedance of a short-circuited line. Referring to Fig. 5-10, it may be interpreted as meaning that Z is the impedance at any point on a line; the line extends

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as are the sine and cosine functions, but since the tangent and cotangent can be computed from the sine and cosine if required, charts are not given here. See Jahnke and Emde, "Tables of Functions with Formulae and Curves" (Ref. 11), and also A. E. Kennelly, "Chart Atlas of Complex Hyperbolic and Circular Functions" (Ref. 12).

from that point to a junction point where it joins another line. The first line has a propagation constant B, a length x, and its contribution to the argument of the tangent function is Bx. The additional line beyond the junction point is terminated in a short circuit. Its contribution to the argument of the tangent function is D. This additional line is really nonexistent; it is merely a hypothetical line of such characteristics that its input impedance is Z_r.

Similar expressions for voltage and current distribution are derived by applying Formula 28 of Table II to Eqs. 4-8 and 4-12. They are

$$V = V_r \sqrt{1 + \left(j \frac{Z_0}{Z_r}\right)^2} \sin (Bx + D) \tag{5-39}$$

$$I = I_r \sqrt{1 + \left(\frac{Z_r}{jZ_0}\right)^2} \cos (Bx + D) \tag{5-40}$$

These equations are greatly helpful for a line terminated in a purely reactive load. If Z_0 is a real number and Z_r is purely imaginary, either positive or negative, Z_r/jZ_0 is real, and D is real.

For example, consider a line with a load that has an impedance Z_r equal to jZ_0. Such a load is inductive reactance. From Eq. 5-38, $\tan D = 1$; therefore $D = \pi/4$. By Eq. 5-39,

$$V = \sqrt{2}\, V_r \sin (Bx + \pi/4) \tag{5-41}$$

This describes a sine-wave distribution of voltage along the line, but with the sine wave slid sideways, to the left, $\frac{1}{8}$ wavelength, corresponding to the $\pi/4$ term. Figure 5-6*f* shows such a voltage distribution. Current distribution on such a line, found from Eq. 5-40, is also plotted in the figure. The impedance is readily determined from Eq. 5-37.

Next consider that the same line has a capacitive load for which $Z_r = -jZ_0$. For this load, $\tan D = -1$, and $D = 3\pi/4$. Three-eighths of a wavelength of short-circuited line is equivalent to such a load, as indicated in Fig. 5-6*g*. Curves of current distribution and impedance can also be drawn.

For any line with a reactive load, the current, voltage, and impedance curves will be displaced sideways more or less depending on the magnitude of the load reactance.

For a line with loss, or a load that consumes energy, B and D are in general complex. Computation then becomes somewhat more involved,[7] but the general principle is the same.

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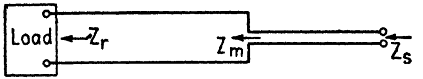
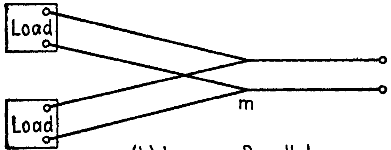
[7] See Sec. 6, Chap. 8, regarding evaluation of inverse trigonometric functions. In evaluating D, there will be ambiguity of sign of the components, and we may be guided by the following criterion (due to P.-K. Tien). Let us write $D = d_1 + jd_2$.

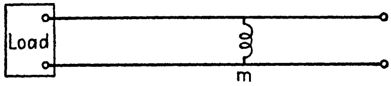
8. Parallel and Series Connections. Frequently in practice a transmission line is not merely connected to a load at one end and to a generator at the other, but is part of a more complicated system. For instance, one section of line may be in series with another section with different characteristic impedance, as in Fig. 5-11*a*. Or a line may divide, as in Fig. 5-11*b*. A line may supply not only one load, but may be connected to other loads along the line, as in Fig. 5-11*c*. One section of a line may supply a subsequent section of the line by being connected to it through an impedance, or through a bad joint in the wire which is effectively a high series resistance, as in Fig. 5-11*d*.



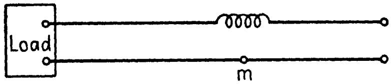
(a) Lines in Series



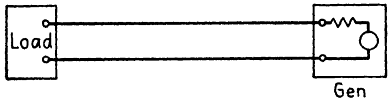
(b) Lines in Parallel



(c) Shunt Impedance



(d) Series Impedance



(e) Generator Impedance

FIG. 5-11. Composite lines.

In each case, an incident wave is partially reflected at the junction, where there is a discontinuity of impedance. Basically, all the examples of Fig. 5-11 are similar.

Let us illustrate the solution by reference to Fig. 5-11*a*, in which a line changes characteristic impedance at point *m*. This may be visualized as a point at which a cable circuit is connected to an open-wire circuit, or as a connection of one coaxial line to another coaxial line of different dimensions. The load impedance Z_r is connected to the end of the left-hand section of line, and from the known value of Z_r the behavior of the line from the load to point *m* is determined. Input impedance to this left-hand section is computed, giving a value that we shall call Z_m. But Z_m is also the terminating impedance for the right-hand section, and when Z_m is known the operation and characteristics of the right-hand section can be found. Thus the study of the operation of the two lines in series may be completed.

Since d_1 and d_2 are analogous to βx and $-\alpha x$, respectively, we may know that for any dissipative load to be represented by D:

d_1 is positive, and is between 0 and $\pi/2$ if the load is inductive, and between $\pi/2$ and π if the load is capacitive (more precisely, between 0 and $\pi/2$ if the imaginary component of Z_r/Z_0 is positive; between $\pi/2$ and π if the imaginary component of Z_r/Z_0 is negative).

d_2 is negative.

The same method is applied to the divided line of Fig. 5-11*b*, as follows. There are two lines to the left of point *m*. The input impedance of each of these, looking into each line at point *m*, is found. These two impedances in parallel constitute the load on the single line that arrives at point *m* from the right.

The input impedance of line and load, looking to the left at point *m* in Fig. 5-11*c*, in parallel with the impedance connected at that point, is the terminating impedance for the right-hand section of line. In Fig. 5-11*d*, the impedance of line and load to the left of point *m* is added in series with the inserted lumped impedance.

Generator impedance is always in series with the transmission line, as in Fig. 5-11*e*. Power, telephone, and radio-frequency lines all have sending-end equipment with internal impedance. If the sending-end network is more complicated than a single impedance, it may be simplified to a single source of constant voltage and a single series impedance by use of Thévenin's theorem.

It often happens that voltage is known, and is held constant, not at the sending-end terminals of the transmission line, but at a point "behind the generator impedance." Sending-end voltage can be found by noting that the generator voltage divides proportionately between the generator impedance and the line input impedance, as between any two impedances in series. Having computed sending-end voltage from this relation, receiving-end voltage and all other line voltages and currents can be determined.

It is entirely possible to write equations for line voltages and currents in terms of generator internal voltage and generator impedance, together with the transmission-line parameters. This is sometimes useful when working with specific problems, and it can readily be done when needed.

9. Balanced Lines. In previous work, the effect of ground has been neglected. With most practical transmission lines, the mere presence of ground does not much affect the operation of the line itself. (Obvious exceptions are one-wire telephone or telegraph lines with ground return and unbalanced three-phase power lines with ground return.) The small change in the line constants that results from the proximity of ground is usually negligible. So, also, is the slight increase of energy loss resulting from current in the earth. However, the presence of ground has effects of practical importance when the mutual influences of one line on another are considered.

Telephone circuits are, in general, carefully "balanced" to ground. This means that the two wires of the line are identical, and when any piece of apparatus is put into the line (such as loading coils, filters, etc.) half the impedance is connected to one wire and half to the other. Neither wire is connected to ground, for the potential of one wire of a balanced line must be equal and opposite to that of the other with respect to ground.

Not only is there no metallic connection to ground, but capacitance to ground must be equalized between the two sides of the circuit. Thus a line consisting of two wires in a horizontal plane is balanced, but if the two wires lie in a vertical plane, the line is not balanced as the lower wire has a greater capacitance to ground. The reason for balancing telephone lines is to maintain equal coupling from each of the two wires of a circuit to other nearby circuits, and thus to prevent mutual inductive and capacitive effects that result in *crosstalk*.

Radio-frequency two-wire lines may be balanced, but terminal equipment sometimes makes balancing impracticable. Coaxial lines are inherently unbalanced. The outer conductor of a coaxial line is commonly connected to ground. Coaxial lines have no electric or magnetic fields outside the outer conductor, and so have no mutual inductance or capacitance with other circuits, and no radiation. A coaxial line is sometimes called *self-shielding*.

Occasionally it is necessary to use a line that is both shielded and balanced. A double coaxial line may then be used, or a balanced shielded pair. These can be directly connected to other balanced lines, whereas a simple coaxial line cannot (see Fig. 12-1).

PROBLEMS

5-1. Using data from Fig. 5-2, plot a vector diagram such as Fig. 5-1 for a telephone line at 1,000 cycles. Relative *angles* of y, z, zy, γ, B, and Z_0 are to be shown; *magnitudes* of vectors need not be comparable.

5-2. Using data from Fig. 5-3, plot a vector diagram such as Fig. 5-1 for a power line at 60 cycles. Relative *angles* of y, z, zy, γ, B, and Z_0 are to be shown; *magnitudes* of vectors need not be comparable.

5-3. A telephone line consists of two 165-mil-diameter copper wires with 12-inch spacing; $r = 4.11$ ohms per mile of two-wire line, and $g = 0$. $K_\varphi = 0$. Compute and plot Z_0 (magnitude and angle), attenuation, and velocity of propagation, as functions of frequency. The line is similar to that of Fig. 5-2 except for size of wire; compare the results. (This type of line was formerly used, but recent practice favors lighter wires.)

5-4. A power line (Tennessee Valley Authority) has conductors with an equivalent spacing (b in Table I) of 20.2 feet. Use 0.99 inches for the *diameter* of the conductor (this is outside diameter of the ACSR cable, an approximation), and $K_\varphi = 1$. Compute l and c to neutral; $r = 0.117$ ohms per mile of conductor, $g = 0$. Compute and plot Z_0 to neutral (magnitude and angle), attenuation, and velocity of propagation, as functions of frequency (10 to 1,000 cycles). Compare with Fig. 5-3 (for a Hoover Dam line).

5-5. Assuming a 10-mile length of the telephone line of Fig. 5-2 is terminated in its characteristic impedance, find the ratio of power input to power output at 50, 200, and 1,000 cycles. (A relation to *decibels* will appear later.)

5-6. Assuming a 10-mile length of the cable circuit of Fig. 5-5 is terminated in its characteristic impedance, find the ratio of power input to power output at 50, 200, and 1,000 cycles. (A relation to *decibels* will appear later.)

5-7. Using data from Fig. 5-5, compute the angles of vectors in Fig. 5-4 for a cable circuit at 1,000 cycles.

5-8. For the telephone line of Fig. 5-2, compute attenuation at 1,000 cycles (*a*) from Eq. 5-5, (*b*) from Eq. 5-6, and (*c*) from Eq. 5-16. Comment on the relative accuracy of the three values.

5-9. A telephone line 100 miles long has 10 volts direct voltage applied at the sending end. The receiving end is open-circuited. Given $r = 10$ ohms and $g = 1$ micromho per mile of circuit, find the receiving-end voltage.

5-10. For the d-c line of Prob. 5-9, find the sending-end current.

5-11. From inspection of Fig. 5-9, what can you tell about the load on the line? Is it inductive, capacitive, or resistive? What is the reflection factor? Is the magnitude of the load greater or less than Z_0, and how much?

5-12. The telephone line of Fig. 5-2, 200 miles long, is open-circuited at the receiving end. Test voltage applied at the sending end is 10 volts at 1,000 cycles. Compute and plot voltage and current along the line.

5-13. The telephone line of Fig. 5-2 is open-circuited at the receiving end. Compute and plot input impedance (magnitude and angle) at 1,000 cycles as a function of length of line (0 to 200 miles).

5-14. The telephone line of Fig. 5-2 is open-circuited at the receiving end. For a 60-mile line, compute and plot input impedance (magnitude and angle) as a function of frequency (100 to 3,500 cycles).

5-15. A coaxial line consists of a wire of 3-millimeter diameter and a tube of 8-millimeter inside diameter. Neglect loss. It is terminated in $R_r = 25$ ohms. Frequency is 250 megacycles. Compute impedance at $x = 10$, 20, and 30 centimeters. Sketch a diagram to indicate (approximately) current, voltage, and impedance variation (as in Fig. 5-8) along 1 wavelength of line. Assume air dielectric.

5-16. Repeat Prob. 5-15 for a line terminated in $R_r = 150$ ohms.

5-17. The line of Prob. 5-15 is terminated in a load capacitance of 20 micromicrofarads. Find D of Eq. 5-37, and sketch impedance as a function of length of line (to 1 wavelength).

5-18. The line of Prob. 5-15 is terminated in $Z_r = 50 + j50$ ohms. Using Eq. 5-39, plot V/V_r as a function of distance (to 1 wavelength).

5-19. A low-loss line contains a lumped resistance, as in Fig. 5-11*d*, that is equal to the characteristic impedance of the line. Sketch (approximately) curves of current and voltage distribution along the line assuming the load impedance equals Z_0.

5-20. All lines in Fig. 5-11*b* have the same Z_0. Assuming that each branch line is terminated in its characteristic impedance, sketch current and voltage distribution along the main line.

5-21. Which of the five lines of Fig. 12-1 are balanced to ground? Tell whether balance is (*a*) impossible, or (*b*) possible but not inherent, or (*c*) inherent for that type of line.

CHAPTER 6

LINE CONSTANTS: INDUCTANCE AND CAPACITANCE

1. Computation from Dimensions. The four basic line constants, inductance, capacitance, resistance, and conductance per unit length, were defined and discussed in Chap. 1. They were defined in terms of measurements on a transmission line. It was explained that the inductance and capacitance are the same at all frequencies (except for a minor variation of inductance resulting from skin effect at high frequency which will be discussed in Chap. 7). Resistance is constant through a low-frequency range, and at higher frequencies can be computed from the low-frequency value. These simple relations make it possible to compute r, l, and c from the dimensions and the properties of the materials of the transmission line.

2. Inductance. Inductance will first be discussed without considering skin effect. The modification resulting from skin effect will be described in Sec. 9, Chap. 7.

Inductance is a measure of the reactive voltage drop along a transmission line. Current flowing in the line produces a magnetic field about the conductors, and since this magnetic field links with the transmission line it induces a reactive voltage in the line as the line current alternates, and consequently the magnetic flux linkages increase and decrease. The first step in the solution for inductance is to define the magnetic field about a conductor carrying current.

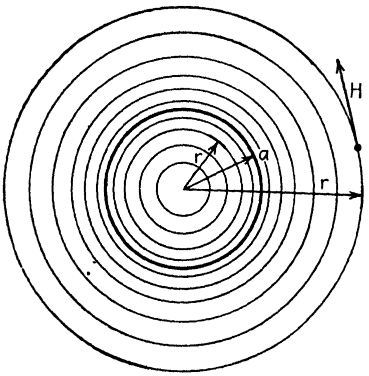


FIG. 6-1. Magnetic field about an isolated round wire.

Consider a single long, straight conductor carrying a steady current. A cross section of the conductor is shown in Fig. 6-1. The current flowing in this conductor must return in some other conductor, but for the present it will be assumed that the other conductor is at such a great distance that it has no appreciable effect on the magnetic field in the vicinity of the conductor shown in the diagram. This simplifies the problem by giving perfect symmetry to the field about the conductor, permitting the magnetic flux lines, as in the figure, to be concentric circles.

The condition that defines the strength of the magnetic field is that the magnetomotive force about any closed path is equal to the current enclosed within that path.[1] Symbolically,

$$\oint \mathbf{H}\cdot d\mathbf{s} = i \tag{6-1}$$

where the integral is taken around a closed path, and the symbols have the following meanings:

$\mathbf{H}$ is the magnetic intensity (ampere-turns per meter)
$\mathbf{s}$ is distance along the path of integration (meters)
i is current enclosed within the path of integration (amperes)

The dot between $\mathbf{H}$ and $d\mathbf{s}$ indicates that only the component of magnetic intensity $\mathbf{H}$ that lies along the path of integration parallel to $d\mathbf{s}$ is to be taken into account.

To find the magnetic field that circles about the conductor of Fig. 6-1, we first consider that there will be symmetry. This means that the magnitude of the magnetic intensity will be the same at every point on a circle of radius r. It is known from other considerations (*i.e.*, that the divergence of the magnetic field is zero) that the direction of the magnetic intensity vector is everywhere tangential to the circle. Hence the field at every point of the circular path of integration has magnitude H and a tangential direction.

This makes it possible to write, for a circular path of integration,

$$\oint H\, ds = i \tag{6-2}$$

and since H is constant along the path of integration it may be taken out from under the integral sign giving

$$H \oint ds = i \tag{6-3}$$

Since the integral of ds is merely the circumference of the circle of radius r,

$$2\pi r H = i \tag{6-4}$$

or

$$H = \frac{i}{2\pi r} \tag{6-5}$$

This, then, is the magnetic field intensity about a circular conductor. In this derivation it is assumed that the path of integration surrounds the conductor and that r is greater than the conductor radius a.

---

[1] See, for instance, "Fundamentals of Electric Waves," Eq. 199, p. 85 (Ref. 29).

If magnetic field is to be determined within the conductor, r must be less than a. Such a circle does not enclose all the current carried by the conductor. The amount of current within such a circle must be found.

In the absence of skin effect, current density is uniform throughout the conductor, and the current within a circle of radius r is proportional to the cross-sectional area of the conductor enclosed within the circle. Hence, if Eq. 6-4 is to be applied to the region within the conductor it must be written

$$2\pi r H = \left(\frac{r}{a}\right)^2 i \tag{6-6}$$

from which

$$H = \frac{1}{2\pi}\frac{r}{a^2} i \tag{6-7}$$

This is the magnetic field intensity within a circular conductor. It will be seen that at the surface of the conductor, where $r = a$, Eqs. 6-5 and 6-7 give identical values. There is no discontinuity in H at the surface of the conductor. Within the conductor, H is directly proportional to radius; outside the conductor, H is inversely proportional to radius. This relation is shown diagrammatically in Fig. 6-2.

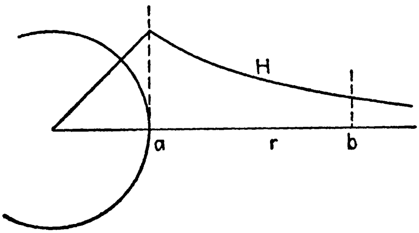


FIG. 6-2. Plot of magnetic intensity.

It is next necessary to find the magnetic flux density B. This is found from Eqs. 6-5 and 6-7, multiplying H by the magnetic permeability μ:

About the conductor:

$$B = \frac{\mu i}{2\pi r} \tag{6-8}$$

Within the conductor:

$$B = \frac{\mu r i}{2\pi a^2} \tag{6-9}$$

In Eq. 6-8, μ is the permeability of the space around the conductor; if this is empty space, $\mu = 4\pi \times 10^{-7}$, and the same value is satisfactory for air or any insulating material. In Eq. 6-9 the permeability of the conductor itself is to be used for μ; for copper, brass, aluminum, or any nonmagnetic material, the value is the same as for empty space.' The only important deviation in the value of μ is for a conductor of iron, steel, or some ferromagnetic alloy; the value for μ in Eq. 6-9 is then $4\pi \times 10^{-7}$ multiplied by the *relative* permeability of the material. B is in webers per square meter.

The next step is to translate magnetic field into inductance. Inductance may be defined as

$$L = \frac{N\Phi}{i} \tag{6-10}$$

where L is inductance (henrys) of a circuit
$N\Phi$ is flux linkages (weber-turns) of magnetic field with the circuit
i is the current (amperes) in the circuit that produces the magnetic field

We call the inductance of unit length of a transmission line l. To compute l, the flux linkages of the magnetic field with the transmission-line conductors are to be found. The first step is to find how many flux lines surround the conductor of Fig. 6-1 or 6-2. Considering flux external to the conductor, flux density is given by Eq. 6-8; the flux surrounding the conductor, between the surface at radius a and any greater radius b, is found by integrating flux density between limits a and b. For unit length of conductor (1 meter),

$$\Phi = \int_a^b \frac{\mu i}{2\pi r}\,dr = \frac{\mu i}{2\pi} \ln \frac{b}{a} \tag{6-11}$$

All flux external to the conductor links all the current of the conductor once and only once; therefore N in Eq. 6-10 is 1. Equation 6-11 therefore gives $N\Phi$ as well as Φ.

Considering the flux within the conductor, flux density is given by Eq. 6-9. For flux within the conductor, N is not 1. A flux line within the conductor encloses only part of the total current, and is said to produce a partial linkage. If the radius of the flux line is r, and current is uniformly distributed within the conductor, the fraction of total current enclosed within the flux line is $(r/a)^2$. This fraction is the value of N for a flux line of radius r within the conductor. Hence the flux linkages between the center of the conductor and the surface are found by integrating $(r/a)^2\Phi$ between the limits 0 and a. Using Eq. 6-9,

$$N\Phi = \int_0^a \left(\frac{r}{a}\right)^2 \frac{\mu r i}{2\pi a^2}\,dr = \frac{\mu i}{8\pi} \tag{6-12}$$

(This use of "partial flux linkages" is discussed in footnote 6, Chap. 7.)

Adding Eqs. 6-11 and 6-12, we obtain the total linkages (weber-turns per meter) of the magnetic field in and about a long, straight, isolated conductor of circular cross section with the current in the conductor which produces that field:

$$N\Phi = \left(\frac{\mu}{2\pi}\ln\frac{b}{a} + \frac{\mu}{8\pi}\right)i \tag{6-13}$$

Since the long, straight conductor carries current, it is part of a circuit, and the flux linkages of Eq. 6-13 contribute to the inductance of that circuit. To complete the solution for inductance it is necessary to know about the other conductors of the circuit. We must find all the flux linkages produced by current in all the conductors, and the result, in Eq. 6-10, gives the inductance.

3. Coaxial Line. The simplest example is the coaxial line, indicated in cross section in Fig. 6-3. Current is carried by the inner conductor, and equal and opposite current is carried by the outer tubular conductor. It is often most convenient to consider the magnetic field from the point of view of superposition: current in the inner conductor produces a circular magnetic field directed, let us say, clockwise, and extending from the inner conductor to infinity; current in the outer conductor produces a circular magnetic field directed counterclockwise, extending from the outer conductor to infinity. Since the currents in the inner and outer conductors are equal in magnitude they produce equal but opposite magnetic fields in all the region beyond the outer conductor, with a consequent resultant field strength of zero. Within the inner surface of the outer conductor, the field is determined entirely by the current in the inner conductor.

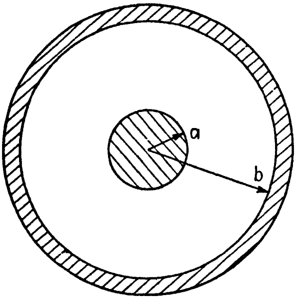


FIG. 6-3. Cross section of a coaxial line.

From another point of view, which is simpler for the coaxial line, there is no magnetic flux surrounding both inner and outer conductors because any path encircling both conductors encloses zero total current.

From either point of view, flux linkages of the magnetic field with the circuit result from flux that is enclosed within the outer conductor, and these linkages are given by Eq. 6-13 if the radius of the inner conductor is a and the inner radius of the outer conductor is b. This expression includes (1) flux within the inner conductor and (2) flux between the conductors. Flux also exists (3) in the material of the outer conductor; it is usual, however, to neglect linkages resulting from flux in this third region, and this is a reasonable approximation because the outer wall is usually thin, because flux density is low at this relatively large radius, and because skin effect that is associated with high frequencies tends to concentrate current at the inner surface of the outer conductor. At very low frequency it might be more exact to let b be the mean radius of the tubular outer conductor, but at high frequency the best approximation results from using the inside radius for b. For any ordinary coaxial line the practical difference is negligible.

Finally, therefore, using linkages from Eq. 6-13 in Eq. 6-10 for inductance, the inductance of unit length of the *coaxial line* is

$$l = \frac{\mu}{2\pi} \ln \frac{b}{a} + \frac{\mu}{8\pi} \tag{6-14}$$

Let us specify that permeability of the region between conductors is $4\pi \times 10^{-7}$ as it is for empty space and for all nonmagnetic materials, and represent the relative permeability of the conductor material by μ_r which will differ from 1 only for iron and ferrous alloys. Then

$$l = \left(2 \ln \frac{b}{a} + \frac{1}{2}\mu_r\right)10^{-7} \tag{6-15}$$

Formula 7 of Table I follows immediately when the common logarithm is substituted for the natural logarithm, using

$$\ln_e x = \ln_e 10 \cdot \log_{10} x = 2.303 \log_{10} x \tag{6-16}$$

(K_φ, the permeability and skin-effect constant in Formula 7, contains μ_r as a factor, and is also affected by frequency, etc., as will be explained in the discussion of skin effect.)

4. Two-wire Line. Consider the two wires shown in Fig. 6-4. Current in the left-hand conductor produces flux that is circular about that conductor, and current in the right-hand conductor produces flux that is circular about the right-hand conductor. The resultant magnetic field has a familiar form, but we need not be concerned with the geometry of the total field, as the inductance of the line is most readily obtained by superposition, as follows.

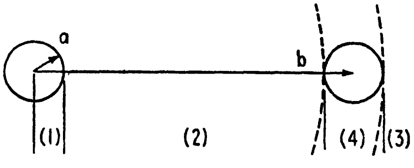


FIG. 6-4. Cross section of a two-wire line.

Current in the left-hand conductor produces flux (1) within the left-hand conductor, and also produces flux (2) in the region between the conductors. In addition, it produces flux (3) that circles around both conductors; but this flux, with radius greater than the spacing between wires, is of no interest, for it does not link with the circuit. A small part of the flux that is produced by current in the left-hand conductor passes *through* the metal of the right-hand conductor, and this flux (4) links with part of the current of the circuit to produce partial linkages, but as an approximation let us assume (for this purpose) that current in the right-hand conductor is concentrated at the center of the conductor. If this assumption is made, flux lines with radius less than b (but greater than a) link fully with the circuit and flux lines with radius greater than b produce no linkages at all. Total linkages produced by current in the left-hand conductor are then obtained from Eq. 6-13, with b the distance between conductor centers.

Current in the right-hand conductor also produces linkages. If the two conductors are the same size and carry the same current, both produce the same number of linkages. Total linkages with the circuit, therefore, are found by multiplying Eq. 6-13 by 2. We then divide flux linkages by current to find the inductance per meter of two-wire line. Using $\mu = 4\pi \times 10^{-7}$ for the permeability of space between the conductors, and a relative permeability of μ_r for the conductor material,

$$l = \frac{\mu}{\pi} \ln \frac{b}{a} + \frac{\mu}{4\pi} = \left(4 \ln \frac{b}{a} + \mu_r\right) 10^{-7} \tag{6-17}$$

Changing from the natural to the common logarithm then gives Formula 3 of Table I for a *two-wire line*:

$$l = \left(9.212 \log_{10} \frac{b}{a} + \mu_r\right) 10^{-7} \tag{6-17a}$$

By this derivation, K_φ of Formula 3 is identified with the relative permeability μ_r of Eq. 6-17*a*. In the next chapter, a more complete interpretation of K_φ will be given.

Formula 1 of Table I, also for a two-wire line, is obtained from Formula 3 by a mere change of units of length. Formula 2, however, for inductance *to neutral*, introduces a new concept. It suggests that we can find the inductance of part of a circuit. This requires more careful consideration of the meaning of inductance.

Fundamentally, inductance relates a voltage to a rate of change of current. Where there is induced voltage, there is inductance. This may be expressed by

$$v = L \frac{di}{dt} \tag{6-18}$$

where L is the inductance of a circuit in henrys, di/dt is the rate of change of current in that circuit in amperes per second, and v is the voltage at the terminals to produce that much rate of change of current.[2]

---

[2] Equation 6-18 is a better definition of L than is Eq. 6-10. They are related by noting that induced voltage equals the time rate of change of flux linkages; that is, $v = d(N\Phi)/dt$. Since, by Eq. 6-10, $N\Phi = Li$, $v = d(Li)/dt$, and if L is constant $v = L\,di/dt$. If L is not constant, it is necessary to abandon, or at least modify, Eq. 6-10.

Whenever a question arises as to what inductance to use, the answer is found by inquiring what voltage is wanted. Thus Formula 1 of Table I gives the inductance to be used if the voltage required is the total voltage induced in both wires of a two-wire transmission line; whereas Formula 2 gives the inductance for computing the voltage induced in only one wire of a two-wire line.

In Fig. 6-5 a transmission line is shown (an electrically *short* line, so that distributed capacitance need not be considered), and the inductance of this line may be used to illustrate the distinction between Formulas 1 and 2. Inductance from Formula 1 gives the total voltage consumed in a pair of wires, and is equal to $V_s - V_r$. Inductance from Formula 2 gives the voltage drop along one wire. This voltage could be read on a voltmeter connected from end to end of the line, as shown, provided the voltmeter leads were at a very great distance from the transmission line so that the magnetic field about the transmission line would not have any appreciable inductive effect on the instrument leads. This voltage is called the voltage drop *to neutral* because it is the difference between $\frac{1}{2}V_s$, the voltage *to neutral* at the sending end of the line, and $\frac{1}{2}V_r$, the voltage *to neutral* at the receiving end.[3]

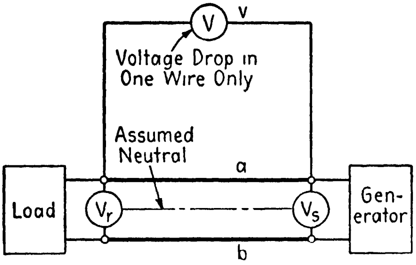


FIG. 6-5. Distinguishing "voltage drop to neutral" from "voltage drop in two-wire line."

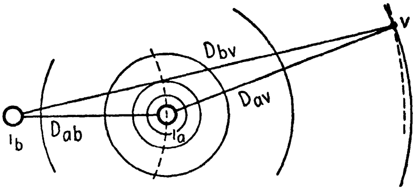


FIG. 6-6. Cross section of a multiconductor system.

5. Flux Linkages in Multiconductor Systems. Another and slightly different derivation of Formula 2 will illustrate a method that is useful in later developments. A cross section of the line of Fig. 6-5 is shown in Fig. 6-6, with a dot on the right indicating the connection to the voltmeter. (This voltmeter lead is not necessarily in the same plane as the two line wires.) It is assumed that this voltmeter is ideal, and does not draw any current; then, the two line currents being equal and opposite,

---

[3] It is unfortunate that two such formulas for inductance must be given, but since both will be encountered in practice it is necessary to understand the distinction. The value given by Formula 1 is here called *inductance per mile of two-wire line*; other common names for the same quantity are *inductance per mile of line* and *inductance per loop mile*. The value given by Formula 2, here called *inductance to neutral per mile of wire* is variously called *inductance per mile of wire*, or *inductance per mile per phase*, or *inductance per mile to neutral*.

$$i_a + i_b = 0 \tag{6-19}$$

To find the inductance of conductor *a* ("one wire to neutral") we first ask to what voltage the inductance is to be related. The answer is: To the induced voltage indicated on the voltmeter of Fig. 6-5, the voltmeter connections being at a very great distance from the transmission-line wires. This is the voltage induced by flux that links the circuit consisting of the line wire *a* and the voltmeter lead *v*. Calling the distance from conductor *a* to *v*, D_{av}; that from conductor *b* to *v*, D_{bv}; and that from conductor *b* to *a*, D_{ab} as in Fig. 6-6, and using R_a for the radius of conductor *a*, we write the following expressions for flux linkages.

Flux linkages produced by the current i_a out to a distance D_{av} are

$$i_a\left(\frac{\mu}{8\pi} + \frac{\mu}{2\pi}\ln\frac{D_{av}}{R_a}\right) \tag{6-20}$$

This is from Eq. 6-13, and assuming the permeability of free space in the region between conductors it can be written

$$i_a\left(\frac{\mu_r}{2} + 2\ln\frac{D_{av}}{R_a}\right)10^{-7} \tag{6-21}$$

Flux linkages produced by the current i_b that link the circuit consisting of conductors *a* and *v* result from flux lines about conductor *b* with radius greater than D_{ab} but less than D_{bv}. Performing the integration of Eq. 6-11 between these limits gives the desired flux linkages produced by i_b:

$$i_b\left(2\ln\frac{D_{bv}}{D_{ab}}\right)10^{-7} \tag{6-22}$$

Adding flux linkages produced by i_a and those produced by i_b gives total flux linkages that induce voltage in the circuit of conductor *a* and the voltmeter:

$$\begin{aligned} N\Phi &= \left(i_a\frac{\mu_r}{2} + 2i_a\ln\frac{D_{av}}{R_a} + 2i_b\ln\frac{D_{bv}}{D_{ab}}\right)10^{-7} \\ &= \left(i_a\frac{\mu_r}{2} + 2i_a\ln\frac{1}{R_a} + 2i_b\ln\frac{1}{D_{ab}}\right. \\ &\qquad \left. + 2i_a\ln D_{av} + 2i_b\ln D_{bv}\right)10^{-7} \end{aligned} \tag{6-23}$$

Noting from Eq. 6-19 that $i_b = -i_a$,

$$N\Phi = \left(i_a\frac{\mu_r}{2} + 2i_a\ln\frac{1}{R_a} + 2i_b\ln\frac{1}{D_{ab}} + 2i_a\ln\frac{D_{av}}{D_{bv}}\right)10^{-7} \tag{6-24}$$

For this derivation to be of value, it is necessary for the voltmeter connection to be so far from the line that it is beyond any significant amount of magnetic field. That is, D_{av} and D_{bv} must be very much greater than D_{ab}. Mathematically, they should both be allowed to increase without limit. The ratio D_{av}/D_{bv} then approaches 1, the logarithm of this ratio approaches 0, and the expression for flux linkages approaches

$$N\Phi = \left(i_a \frac{\mu_r}{2} + 2i_a \ln \frac{1}{R_a} + 2i_b \ln \frac{1}{D_{ab}}\right)10^{-7} \tag{6-25}$$

To find inductance of conductor a from this expression, $-i_a$ is substituted for i_b and flux linkages are divided by current, giving

$$l = \left(\frac{\mu_r}{2} + 2 \ln \frac{1}{R_a} - 2 \ln \frac{1}{D_{ab}}\right)10^{-7} = \left(\frac{\mu_r}{2} + 2 \ln \frac{D_{ab}}{R_a}\right)10^{-7} \tag{6-26}$$

where μ_r is relative permeability of the material of conductor a

R_a is radius of conductor a (any unit of length, but D_{ab} must be measured in the same unit)

D_{ab} is spacing between wires a and b, center to center

l is inductance of conductor a (henrys per meter)

With minor changes of notation, substitution of the common for the natural logarithm, and multiplication by the appropriate constant to give henrys per mile, this becomes Formula 2 of Table I.

The most interesting aspect of Eq. 6-25 is that it illustrates a form of expression that can be extended to apply to transmission lines with any number of conductors. Equation 6-25 and Fig. 6-6 relate to a line of two wires, a and b. If the line has three wires, a similar derivation yields an expression with another logarithmic term added to the equation. In general, if a line has n conductors, the expression for flux linkages that induce voltage in conductor a is

$$(N\Phi)_a = \left(i_a \frac{\mu_r}{2} + 2i_a \ln \frac{1}{R_a} + 2i_b \ln \frac{1}{D_{ab}} + 2i_c \ln \frac{1}{D_{ac}} + \cdots + 2i_n \ln \frac{1}{D_{an}}\right)10^{-7} \tag{6-27}$$

Equation 6-19 is replaced by the general form:

$$i_a + i_b + i_c + \cdots + i_n = 0 \tag{6-28}$$

The derivation of this general expression of Eq. 6-27 is similar to that of Eq. 6-25, except for additional complications in the algebra.[4]

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[4] See, for instance, L. F. Woodruff, "Principles of Electric Power Transmission," Chap. II (Ref. 38).

6. Four-wire Line. To show the use of this general expression for flux linkages, consider a line of four wires arranged as in Fig. 6-7. Such a transmission line is frequently used to supply the antenna of a radio transmitter. Wires a_1 and a_2 are connected together at both ends and comprise, actually, a single conductor. Wires b_1 and b_2 are similarly connected together at both ends and comprise the other conductor of the line.

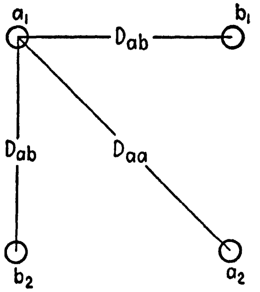

FIG. 6-7. Cross section of a four-wire line.

Wires a_1 and a_2 divide between them the line current and, because the arrangement of the four wires is symmetrical, the current divides equally between the two wires. The other two wires share the returning current, so that the current in each of wires a_1 and a_2 is $\frac{1}{2}i_a$ and the current in each of wires b_1 and b_2 is $-\frac{1}{2}i_a$.

Flux linkages with one of the wires can readily be found by direct application of Eq. 6-27. For wire a_1, with radius R_a,

$$
\begin{aligned}
(N\Phi)_{a1} &= \left(i_{a1}\frac{\mu_r}{2} + 2i_{a1}\ln\frac{1}{R_a} + 2i_{a2}\ln\frac{1}{D_{aa}} + 2i_{b1}\ln\frac{1}{D_{ab}}\right.\\
&\qquad\qquad \left. + 2i_{b2}\ln\frac{1}{D_{ab}}\right)10^{-7}\\
&= i_a\left(\frac{\mu_r}{4} + \ln\frac{1}{R_a} + \ln\frac{1}{D_{aa}} - \ln\frac{1}{D_{ab}} - \ln\frac{1}{D_{ab}}\right)10^{-7}\\
&= i_a\left(\frac{\mu_r}{4} + \ln\frac{D_{ab}^{\,2}}{R_aD_{aa}}\right)10^{-7}\\
&= i_a\left(\frac{\mu_r}{4} + \ln\frac{D_{ab}}{R_a\sqrt{2}}\right)10^{-7} \qquad (6\text{-}29)
\end{aligned}
$$

To find inductance of the two wires a_1 and a_2 in parallel, divide by their total current i_a:

$$l_a = \left(\frac{\mu_r}{4} + \ln\frac{D_{ab}}{\sqrt{2}\,R_a}\right)10^{-7} \qquad (6\text{-}30)$$

It is now necessary to consider exactly what inductance this is. It is the inductance (in henrys per meter) that gives voltage drop in either of the pair of wires a_1 and a_2, using the total current of those two wires, i_a. There is an equal voltage drop in the pair of wires b_1 and b_2; therefore the total inductance *per loop-meter* of the four-wire line is twice the value of Eq. 6-30:

$$l = \left(\frac{\mu_r}{2} + 2\ln\frac{D_{ab}}{\sqrt{2}\,R_a}\right)10^{-7} \qquad (6\text{-}31)$$

It is this doubled value that would be measured on a bridge applied to one end of the four-wire line with the other end short-circuited, and this is the value that would commonly be used for finding characteristic impedance and other transmission-line parameters.

This general method is readily applicable to multiconductor transmission lines if symmetry assures equal division of current among the various wires. If division of current is unequal, another consideration must be included.

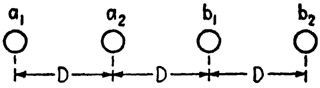

FIG. 6-8. Cross section of a four-wire line.

Unequal Division of Current. Figure 6-8 shows a four-wire line with different configuration. A practical example of such a line is a telephone phantom circuit. As in the previous example, the lines a_1 and a_2 share the total current, as do b_1 and b_2 also, but in this case they do not share the current equally. Because of the lack of symmetry in arrangement of the wires, the inductive reactance of wire a_1 is greater than that of wire a_2.

To find the inductance of such a line, it is first necessary to find the division of current between the wires. What determines the division of current? The determining factor is this: since a_1 and a_2 are connected together at both ends, the total voltage drop along a_1 must be the same as the total voltage drop along a_2. This total voltage drop includes both the drop in the line resistance and the inductive reactance drop. Thus, using rms values of alternating current and voltage,

$$\begin{aligned} V_{a1} &= r_{a1}I_{a1} + jx_{a1}I_{a1} \\ &= V_{a2} = r_{a2}I_{a2} + jx_{a2}I_{a2} \end{aligned} \tag{6-32}$$

In these equations, V is voltage drop per meter of length of wire if r and x are resistance and reactance, respectively, per meter. Introducing inductance,

$$r_{a1}I_{a1} + j\omega l_{a1}I_{a1} = r_{a2}I_{a2} + j\omega l_{a2}I_{a2} \tag{6-33}$$

or

$$r_{a1}I_{a1} + j\omega(N\Phi)_{a1} = r_{a2}I_{a2} + j\omega(N\Phi)_{a2} \tag{6-34}$$

We now use Eq. 6-27 to write expressions for the flux linkages. The radius of each wire being R,

$$\begin{aligned} (N\Phi)_{a1} = \Big(I_{a1}\frac{\mu_r}{2} + 2I_{a1}\ln\frac{1}{R} + 2I_{a2}\ln\frac{1}{D} + 2I_{b1}\ln\frac{1}{2D} \\ + 2I_{b2}\ln\frac{1}{3D}\Big)10^{-7} \end{aligned} \tag{6-35}$$

Symmetry is such as to require that $I_{a1} = -I_{b2}$ and $I_{a2} = -I_{b1}$; so Eq. 6-35 simplifies to

$$(N\Phi)_{a1} = \left[I_{a1}\left(\frac{\mu_r}{2} + 2\ln\frac{3D}{R}\right) + I_{a2}(2\ln 2)\right]10^{-7} \tag{6-36}$$

Similarly,

$$\begin{aligned}(N\Phi)_{a2} &= \left(I_{a2}\frac{\mu_r}{2} + 2I_{a2}\ln\frac{1}{R} + 2I_{a1}\ln\frac{1}{D} + 2I_{b1}\ln\frac{1}{D}\right.\\ &\qquad\qquad\qquad \left. + 2I_{b2}\ln\frac{1}{2D}\right)10^{-7}\\ &= \left[I_{a2}\left(\frac{\mu_r}{2} + 2\ln\frac{D}{R}\right) + I_{a1}(2\ln 2)\right]10^{-7}\end{aligned} \tag{6-37}$$

When Eqs. 6-36 and 6-37 are substituted into Eq. 6-34, there results an equation in which the only unknowns are I_{a1} and I_{a2}. Solution of this equation therefore gives either one of the currents in terms of the other, or we may solve for both in terms of the total current I_a, knowing that $I_a = I_{a1} + I_{a2}$. The result of the latter solution is

$$I_{a1} = I_a\left[\frac{1}{2} - \frac{j\omega \ln 3}{2r_a \times 10^7 + j\omega\left(\mu_r + 2\ln\frac{3D^2}{4R^2}\right)}\right] \tag{6-38}$$

$$I_{a2} = I_a\left[\frac{1}{2} + \frac{j\omega \ln 3}{2r_a \times 10^7 + j\omega\left(\mu_r + 2\ln\frac{3D^2}{4R^2}\right)}\right] \tag{6-39}$$

This unequal division of current is specially interesting because it is an introduction to the general subject of skin effect. Proximity effect is another related phenomenon. The essential point is this: when there are two or more paths along which current can travel in parallel, it is necessary that the voltage drop be the same along both or all. If the flux pattern is such that inductance of one path is less than inductance of another, more current flows in the path with less inductance. Referring to Fig. 6-8, the circuit a_2b_1 has less inductance than the circuit a_1b_2 because of its closer spacing. It therefore carries greater current. In general, an alternating current in a conductor tries to flow as far as possible from other currents flowing in the same direction, and as close as possible to the oppositely directed current in the return conductors; this is an oversimplified statement, but it directs attention to the essential factors.[5]

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[5] This effect is not to be confused with the physical force exerted on conductors carrying current in a magnetic field, which is an entirely different phenomenon and appears with unidirectional as much as with alternating current.

Referring to Eqs. 6-38 and 6-39, it will be seen that at zero frequency the current divides equally, for ω is zero. At high frequency the current in the outer conductor is less than the current in the inner conductor. With this unequal distribution of current, the loss of power in the line resistance is greater than it would be if current were equally divided between the two conductors, for the power loss is proportional to the square of current in each conductor.

It is clear that the resistance of such a line will not be found from the resistance of the individual conductors until the division of current is known, nor will the line reactance be found from the individual inductances. Indeed, new definitions of resistance and reactance are needed: the impedance of a line that has unequal current distribution is the ratio of voltage drop to current, and the resistance and reactance of the line are the real and imaginary components, respectively, of this impedance. This definition is adopted because the quantities so defined can be used in almost all problems as are ordinary resistance and reactance. These quantities are often called the *effective alternating-current resistance and reactance* of the line.

For the line of Fig. 6-8, impedance, effective resistance, and inductance can be found by the above procedure. The general expressions are rather complicated, but the answers to numerical problems are readily found. Equations 6-38 and 6-39 are substituted into Eq. 6-36, and that into one side of Eq. 6-34 to find the voltage drop, which is then divided by I_a to give impedance. Effective a-c resistance, which is the real part of the impedance, is always greater than half the d-c resistance of one wire alone; this results from the unequal distribution of current, and the effective a-c resistance increases with frequency.

Transposition. As a practical matter, it is wasteful to permit the resistance of a transmission line to be greater than necessary, and telephone phantom circuits as well as other lines using the wire configuration of Fig. 6-8 actually avoid an increase of effective resistance by transposing the wires. This means that the wire called a_1 occupies the outer position on a crossarm, as indicated in Fig. 6-9, for half the total length of the line, and it is exchanged with a_2 to occupy the inner position for the other half of the length. The result of transposition is to provide the same total

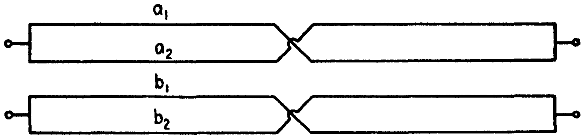

FIG. 6-9. Transposition of conductors.

flux linkages and hence the same total induced voltage in the two wires a_1 and a_2 when the entire length is considered, and with equal induced voltage there is equal division of the current.

If there is transposition, the determination of inductance is relatively simple. Flux linkages per meter are found by averaging Eqs. 6-36 and 6-37, with $I_{a1} = I_{a2}$. The inductance of the two wires in parallel is found by dividing by the total current I_a:

$$l_a = \left(\frac{\mu_r}{4} + \ln \frac{2\sqrt{3}D}{R}\right)10^{-7} \tag{6-40}$$

this being inductance in henrys per meter to neutral. Henrys per loop-meter are found by multiplying Eq. 6-40 by 2.

7. Geometric Mean Distance. It is interesting to generalize the examples that have been discussed and to consider the inductance of a group of n conductors that share current equally. These conductors may be arranged in any configuration, and we shall identify them by letters from a to n (see Fig. 6-10). The current in each conductor is I/n. The return current is carried by another group of n conductors, lettered from a' to n'; the total return current must be equal but opposite to the total outgoing current, and the current in each return conductor must be $-I/n$. From Eq. 6-27, flux linkages about conductor a are

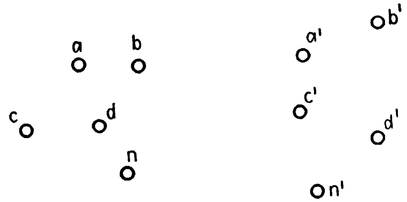

FIG. 6-10. Cross section of two groups of conductors.

$$(N\Phi)_a = \frac{I}{n}\left(\frac{\mu_r}{2} + 2\ln\frac{1}{R_a} + 2\ln\frac{1}{D_{ab}} + 2\ln\frac{1}{D_{ac}} + \cdots + 2\ln\frac{1}{D_{an}}\right.$$
$$- 2\ln\frac{1}{D_{aa'}} - 2\ln\frac{1}{D_{ab'}} - 2\ln\frac{1}{D_{ac'}} - \cdots$$
$$\left. - 2\ln\frac{1}{D_{an'}}\right)10^{-7} \tag{6-41}$$

We wish to find inductance; the usual question arises: Exactly what inductance is desired? We wish to find the inductance that will give the voltage drop in conductor a when multiplied by the rate of change of the total current I in the whole group of conductors. Therefore we divide the flux linkages about conductor a by the total current I:

$$l_a = \frac{1}{n}\left(\frac{\mu_r}{2} + 2\ln\frac{1}{R_aD_{ab}D_{ac}\cdots D_{an}}\right.$$
$$\left. + 2\ln D_{aa'}D_{ab'}D_{ac'}\cdots D_{an'}\right)10^{-7}$$

$$= \left[\frac{2}{n}\left(\frac{\mu_r}{4} + \ln \frac{1}{R_a D_{ab} D_{ac} \cdots D_{an}}\right) + 2 \ln \sqrt[n]{D_{aa'} D_{ab'} D_{ac'} \cdots D_{an'}}\right] 10^{-7} \qquad (6\text{-}42)$$

The second logarithmic term in this expression is recognizable as the geometric mean of the distances from conductor a to each of the conductors of the return group. The first logarithmic term can be put in similar form as follows: first, assume that the permeability of the conductor material is the same as that of empty space, so that $\mu_r = 1$; second, note that

$$\frac{1}{4} = \ln_e e^{\frac{1}{4}} = \ln \frac{1}{e^{-\frac{1}{4}}} \qquad (6\text{-}43)$$

Then

$$l_a = 2 \ln \frac{\sqrt[n]{D_{aa'} D_{ab'} D_{ac'} \cdots D_{an'}}}{\sqrt[n]{e^{-\frac{1}{4}} R_a D_{ab} D_{ac} \cdots D_{an}}} 10^{-7} \qquad (6\text{-}44)$$

It is now helpful to introduce a mathematical concept known as *geometric mean distance.* The geometric mean distance from one point to a set of n points is the geometric mean of the n distances; that is, it is the nth root of the product of the n distances. The geometric mean distance from one set of n points to another set of n' points is the geometric mean of all the nn' distances. It is not necessary that the two sets of points should be different, for we may find the geometric mean of the distances between every pair of points in the same set.

The geometric mean distance between two *areas* is also defined. It is found by dividing each area into many equal elementary areas and finding the geometric mean of all the distances between pairs of elements; the result approaches the *geometric mean distance between the two areas* as the number of elements in each area is increased without limit.[6] It is not

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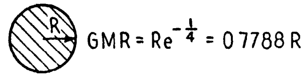
[6] Such a limit is best expressed as an integral. J. C. Maxwell, in "A Treatise on Electricity and Magnetism," Art. 691 (Ref. 17), gives for the geometric mean distance, indicated by GMD,

$$A_1 A_2 \ln \text{GMD} = \iiiint \ln r \, dx \, dy \, dx' \, dy' \qquad (6\text{-}45)$$

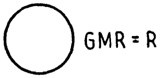
where $dx\,dy$ is an element of area of the first section, $dx'\,dy'$ an element of area of the second section, and r the distance between these elements, the integration being extended first over every element of the first section, and then over every element of the second.

necessary for the two areas to be distinct, and the geometric mean of the distances between all pairs of points in the same area can be found; this is often called the *geometric mean radius of the area.* These quantities are commonly abbreviated GMD and GMR, respectively.

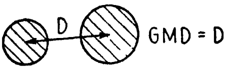
GMD and GMR are mathematical quantities, having no necessary relation to electricity. Purely mathematical analysis will show, for instance, that the geometric mean distance between one circular area and another circular area is equal to the actual linear distance between their centers; this is true for circular areas of any size, provided they do not overlap (see Fig. 6-11*c*). This theorem, which we shall not attempt to prove, will be useful in the next few pages.



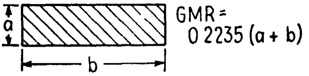
(a) Circular Area



(b) Circular Line



(c) Two Circular Areas



(d) Rectangular Area

FIG. 6-11. Geometric mean radii and distances.

Another similar theorem is that the geometric mean radius of a circular area is equal to the actual radius of the circular area multiplied by $e^{-\frac{1}{4}}$. As indicated in Fig. 6-11*a*, the geometric mean radius of a circular area is slightly less than eight-tenths of the radius of the circle.

Another theorem is that the geometric mean radius of a circular line (this includes only points on the line in Fig. 6-11*b*, and not points in the enclosed area) is equal to the radius of the circle. There are other theorems relating to various geometric shapes, but these three are enough for many purposes. (Equation 6-54 gives another. Also see the reference to Woodruff in footnote 4 and to Maxwell in footnote 6.)

Returning to Eq. 6-44, it will now be seen that the denominator contains a quantity that is equal to the geometric mean radius of conductor *a*, and

$$l_a = 2 \ln \frac{\sqrt[n]{D_{aa'}D_{ab'}D_{ac'}\cdots D_{an'}}}{\sqrt[n]{(\mathrm{GMR})_a D_{ab}D_{ac}\cdots D_{an}}}\,10^{-7} \qquad (6\text{-}46)$$

This equation can be used to find inductance of multiconductor systems, provided the current is known to divide equally among the conductors either because of the geometry or as a result of transposition. For example, when applied to the transmission line of Fig. 6-7 it gives

$$l_a = 2 \ln \frac{\sqrt[2]{D_{ab}{}^2}}{\sqrt[2]{(\mathrm{GMR})_a D_{aa}}}\,10^{-7} = \ln \frac{D_{ab}}{(\mathrm{GMR})_a \sqrt{2}}\,10^{-7} \qquad (6\text{-}47)$$

which may be compared with Eq. 6-30. When applied to the transposed line of Figs. 6-8 and 6-9 it must be handled in two parts. Inductance of a wire in the position a_1 is

$$l_{a1} = 2 \ln \frac{\sqrt[2]{2D\cdot 3D}}{\sqrt[2]{(\mathrm{GMR})_a\cdot D}}\, 10^{-7} = \ln \frac{6D}{(\mathrm{GMR})_a}\, 10^{-7} \qquad (6\text{-}48)$$

Inductance when the wire is in the position a_2 is similarly found to be

$$l_{a1} = 2 \ln \frac{\sqrt[2]{D\cdot 2D}}{\sqrt[2]{(\mathrm{GMR})_a\cdot D}}\, 10^{-7} \qquad (6\text{-}49)$$

Inductance of the pair of wires a_1 and a_2 of the transposed line is the *average*, for each position is occupied through half the length of the line:

$$\begin{aligned} l_a &= \frac{1}{2}\left(2 \ln \frac{\sqrt{2D\cdot 3D}}{\sqrt{(\mathrm{GMR})_a\cdot D}} + 2 \ln \frac{\sqrt{D\cdot 2D}}{\sqrt{(\mathrm{GMR})_a\cdot D}}\right)10^{-7} \\ &= 2 \ln \frac{\sqrt[4]{D\cdot 2D\cdot 2D\cdot 3D}}{\sqrt[4]{(\mathrm{GMR})_a\cdot(\mathrm{GMR})_a\cdot D\cdot D}}\, 10^{-7} = \ln \frac{2\sqrt{3}D}{(\mathrm{GMR})_a}\, 10^{-7} \qquad (6\text{-}50) \end{aligned}$$

This will be recognized as being the same as Eq. 6-40.

The next to last form of Eq. 6-50 is written in full to show that the numerator is the geometric mean of all possible distances between the conductors of the outgoing or a group and the conductors of the returning or b group. Since there are two conductors in each group, this is the geometric mean of four distances. If there were n conductors in each group, this would be the geometric mean of n^2 distances, distance being measured to all the n conductors in group b from each of n conductors in group a. Similarly, the denominator is the geometric mean of the GMR of one wire, the GMR of the other wire, the distance from wire a to wire b, and the distance from wire b to wire a. Generalizing, this is the n^2 root of the geometric mean radii of the n conductors times the $n(n - 1)$ distances from each conductor to each other conductor. (Each such distance will always appear twice, as measured from conductor j to conductor k, and also from conductor k to conductor j.)

If GMD is used to designate the geometric mean distance from the group of conductors for which inductance is being determined to the other conductors of the transmission line, and GMR is used to designate the geometric mean of the individual geometric mean radii of the conductors of the particular group together with the wire-to-wire distances among the conductors of that group, we can write the general formula:

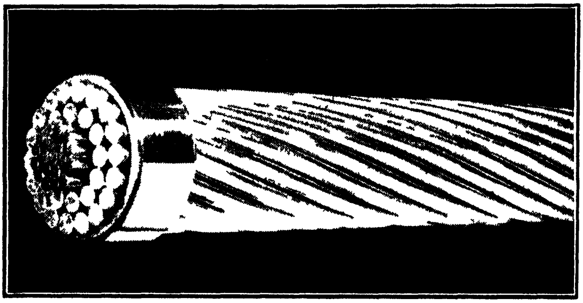
$$l = 2 \ln \frac{\mathrm{GMD}}{\mathrm{GMR}}\, 10^{-7} = 4.606 \log_{10} \frac{\mathrm{GMD}}{\mathrm{GMR}}\, 10^{-7} \qquad (6\text{-}51)$$

This is the inductance that is to be multiplied by the rate of change of *current in all the conductors of the group* to get the voltage drop per meter in that group of parallel-connected conductors. It applies when current divides equally among the conductors.

Equation 6-51 is derived from 6-46 by assuming transposition of the n conductors that carry current in parallel, and averaging the n resulting inductances. As seen in the above example, the arithmetic mean of the logarithms is transformed into the geometric mean of the distances, and Eq. 6-51 results.

This equation is a general guide in computing inductance of multiple-conductor systems of almost any arrangement, provided there is equal division of current. If any problem regarding inductance is found to be puzzling, Eq. 6-51 should be considered as a possible indication of the way to attack the problem.

8. Stranded Cable. It is common for the conductor of a power transmission line to be a cable of many strands of wire with the outer layers spiraled around the inner layers Such a stranded cable is more flexible than a solid rod with the same cross section of copper or aluminum and can be wound on reels for shipment; this greatly aids erection of the lines. The strands are not insulated from each other. The number of strands depends on the size of the conductor and on the flexibility required, and common numbers are 3, 7, 19, 37, 61, or more The cross section of a 7-strand conductor is shown in Fig. 6-12; a 19-strand conductor has another layer of strands, and so on. Alternate layers are spiraled oppositely, to prevent untwisting.



Aluminum cable, steel reinforced The outer aluminum strands carry most of the current; the inner steel strands are for tensile strength Alternate layers of strands are spiraled in opposite directions (The band about the specimen is merely to prevent unraveling) (*Courtesy of Aluminum Company of America*)

If the conducting material is copper, it is usual for all strands to be copper. If the conducting material is aluminum, it is usual for the central strands to be galvanized steel for strength and the outer strands aluminum for electrical conductivity; there may be one central steel strand, or a central group of seven steel strands, or even as many as nineteen strands may be steel in a large cable. Such cable is known as *aluminum cable steel reinforced,* commonly abbreviated ACSR. Various tubular or semitubular forms of conductors are also used for high-voltage power lines to provide the large outside diameter that is necessary to prevent power loss resulting from corona discharge into the air. Also, steel wires are sometimes coated with copper to provide a wire that combines good strength with satisfactory conductivity. The present discussion will be limited to copper cables in which all strands are the same size, and it may be assumed that they divide the total current equally (note that this neglects skin effect), or ACSR cables in which it may be assumed that the aluminum strands share the current equally and the steel strands carry no current.[7]

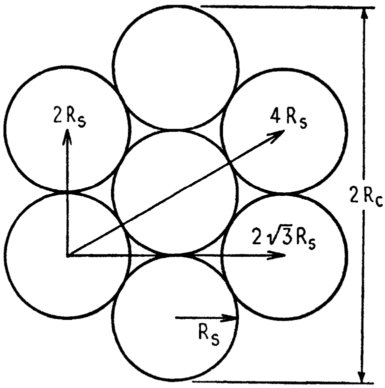
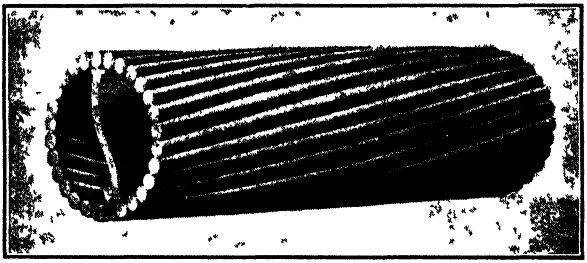


FIG. 6-12. Cross section of a seven-strand cable.



Twisted-I-beam copper power conductor. The core that supports the outer strands has the form of a copper I beam twisted helically This design gives large outside diameter with moderate cross-sectional area. Corona loss is thereby avoided without excessive use of copper. (*Courtesy of Anaconda Wire and Cable Company.*)

Let us find the inductance of a two-conductor line, each conductor of which is a seven-strand cable as shown in Fig. 6-12. Referring to Eq. 6-51, we need the GMD between one cable and the other, and the GMR of one of the seven-strand cables.

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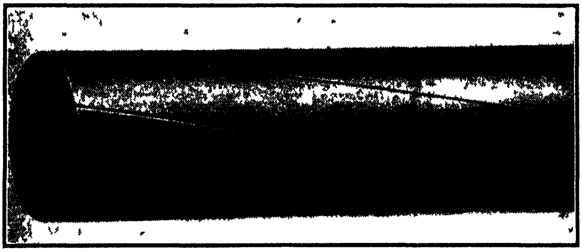
[7] This assumption regarding ACSR conductors is fairly good, but for meticulous work better approximations are available in the literature. See, for instance, Woodruff, "Principles of Electric Power Transmission" (Ref. 38), or the Westinghouse "Reference Book" (Ref. 18).

For all practical purposes, the GMD between one cable and the other is merely the center-to-center distance between cables.

The GMR of one cable is found as follows. There are 7 strands carrying equal current in the same direction, so the GMR will be the forty-ninth root of the product of 7 geometric mean radii and 42 geometric mean distances. The 7 geometric mean radii are those of the 7 strands; the radius of each strand may be called R_s, and the geometric mean radius of each individual strand is $e^{-\frac{1}{4}}R_s$. This quantity appears seven times in the geometric mean for the cable. The 42 distances may be classified according to the strands from which they are measured:

1. The distance from the center strand to each of the six outer strands is $2R_s$. This is the geometric mean distance between strands, for the geometric mean distance between circular areas is the distance between the centers of those areas. This distance appears six times in the computation of the GMR of the cable.



Segmental (Type HH) copper power conductor. This specimen is 1.10 inches in outside diameter and has 500,000 circular mils cross section of copper. The segments are mortised to form a self-supporting flexible hollow conductor. In manufacture, the segments are formed by drawing through dies. (*Courtesy of General Cable Corporation*)

2. From each of the six outer strands, the distance to three adjacent strands is $2R_s$. Hence this distance appears (again) eighteen times in the computation.

3. From each of the six outer strands there are two strands at a distance of $2\sqrt{3}\,R_s$, requiring that this distance be used twelve times.

4. One strand is at a distance of $4R_s$ from each of the outer strands, giving a value that is to be used six times.

Collecting these 49 values we have for the geometric mean radius of the 7-strand cable:

$$\begin{aligned} \text{GMR} &= \sqrt[49]{(e^{-\frac{1}{4}}R_s)^7(2R_s)^6(2R_s)^{18}(2\sqrt{3}\,R_s)^{12}(4R_s)^6} \\ &= R_s \sqrt[49]{(e^{-\frac{1}{4}})^7 2^{48} 3^6} = 2.1767R_s \\ &= 0.7256R_s \qquad \text{(6-52)} \end{aligned}$$

where R_c is the outside radius (that is, half the outside diameter) of the 7-strand cable as in Fig. 6-12.

For cables with other numbers of strands, the computation is similar. The following table gives the relation of GMR to outside radius for cables of various numbers of strands,[8] assuming that all strands in a cable are identical and carry equal current and the strands are laid closely in a compact cable.

TABLE 6-1

GMR OF STRANDED CABLE

Number of Strands	*GMR*
3	$0.677R_c$
7	$0.726R_c$
19	$0.758R_c$
37	$0.768R_c$
61	$0.772R_c$
Solid (or 1 strand)	$0.779R_c$

It is evident that the greater the number of strands, the more closely the current distribution in a stranded cable approximates that in a solid conductor, and this is reflected in the tabulation of GMR's.

9. The General GMR Development. The simple inductance formula of Eq. 6-51 was derived in this chapter through many steps, with a number of approximations and assumptions. The formula, however, is both more general and more rigorous than this development would imply. A direct derivation of this formula can be outlined as follows:

1. A transmission line consists of two conductors of any shape, as suggested in Fig. 6-13 (either or both of these may actually be two or more separate conductors in parallel). The total current through a cross section (such as that of the figure) is zero.

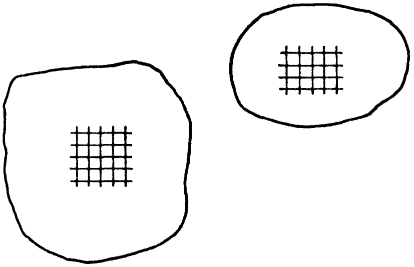


FIG. 6-13. Cross sections showing subdivision of conductors into filamentary elements.

2. Each conductor is considered to be divided into many small filamentary elements, as indicated in the figure. Suppose the left-hand conductor is divided into n elements. Each element carries the same current, I/n. Each element of the other conductor carries the opposite current, $-I/n$.

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[8] Certain of the tabulated GMR's are from "Symmetrical Components," by Wagner and Evans (Ref. 36).

3. An expression for flux linkages about any one of the elements is written. It is assumed that the permeability of the conducting material is that of free space. An average of flux linkages about the *n* different filaments is then found.

4. The number *n* of filamentary elements into which the conductor is divided is now considered to increase without limit. As *n* increases without limit, the expression for average flux linkages approaches a definite limit. If the current density is uniform throughout the conductor, the limit approached by the expression for average flux linkages is

$$2I \ln \frac{\text{GMD}}{\text{GMR}} 10^{-7} \tag{6-53}$$

Dividing by *I* gives the inductance of the conductor. The result is exactly Eq. 6-51, which verifies that expression.

Note that this derivation has the following limitations: it assumes that the material of the conductors is nonmagnetic; and it assumes that current is uniformly distributed through the conductor material. (Yet if these assumptions are untrue, an equivalent GMR can be found by approximation or by experiment, and Eq. 6-53 can still be used.) On the other hand, it is not limited to conductors of any special shape, and computation of inductance is readily extended to any shape for which the GMR can be found. Also, it is not limited to conductors that are far apart relative to their size; whereas Eq. 6-17 was derived as an approximation with the assumption that current was concentrated at the center of the return conductor, it now appears that such an assumption is not necessary and Eq. 6-17 is not an approximation, from that point of view at least, but exact. This would be difficult to prove by any means but the geometric mean distance development.

As an example of the use of this expression for inductance of conductors of unusual shape, consider a transmission line with tubular conductors. It is obviously possible to have a tube with walls of any thickness between the extremely thin-walled tube that is indistinguishable in cross section from a mere circle and the thick-walled tube that approaches a solid rod. Inductance of tubular conductor is found from Eq. 6-51 by using the appropriate GMR. We know that the GMR of the solid rod is 0.779 times the outer radius. The GMR of the thin-walled tube is equal to the radius of the tube (as in Fig. 6-11*b*). The GMR of any other tube must fall between these limits. This fact is an adequate guide in many computations, for the difference between the limiting possibilities is likely to be negligible, but for the sake of greater numerical accuracy Table 6-2 on the following page can be used:

TABLE 6-2

GMR OF TUBULAR CONDUCTOR

Ratio of the Inside Radius of a Tube to the Outside Radius of the Tube	*Ratio of the GMR to the Outside Radius of the Tube*
0.0 (solid)	0.78
0.2	0.79
0.4	0.83
0.6	0.88
0.8	0.94
1.0 (thin wall)	1.00

A tubular transmission line is of practical importance in various uses. The conductors of the high-voltage lines from Hoover Dam to the city of Los Angeles are essentially tubes; they are not smooth on the inside, but the ratio of inside diameter to outside diameter is approximately 0.86. Steel wires coated with copper may be considered as tubular if the conductivity of the steel is neglected and all current is assumed to flow in the copper. At high frequency, skin effect practically eliminates current near the center of a large conductor, with the result that even a solid copper rod acts like a thin-walled tube if the frequency is high (see, in this connection, Sec. 7, Chap. 7).

Another practical form of conductor is of rectangular cross section. As an approximation with a maximum error of about two-tenths of one per cent,

$$\text{GMR of a rectangle} = 0.2235(a + b) \tag{6-54}$$

the sides of the rectangle being a and b. This expression may be used in finding the inductance of bus bars, for instance.

10. Three-phase Inductance. There are three geometrical arrangements of three-phase lines that require consideration.[9] The simplest is a line with the three conductors equally spaced, as at the corners of an equilateral triangle. This is unusual in practice, however, for lines usually have nonequilateral spacing, but the conductors of most lines are systematically transposed and this has the effect of averaging the line inductance among the three conductors. The third possibility is a non-equilateral line that is also nontransposed.

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[9] A related problem is the inductance of the circuit involving the three line wires and ground; the three line wires of a three-phase line, being connected in parallel, constitute one conductor, and the other conductor is the earth. This earth-return circuit is the path of *zero-sequence* components of current in unbalanced operation. It is essentially a single-phase, not a three-phase, problem. Although simple in practice, it is highly complicated in theory. It is of great importance in connection with symmetrical component analysis, and will be left for treatment in books on that subject as, for instance, Wagner and Evans (Ref. 36). However, formulas and tables of zero-sequence quantities are included in Appendix 3 of this book.

The line with *equilateral spacing* is easily handled. A cross section of the line is shown in Fig. 6-14. The distance between conductor centers is D. The radius of each conductor is R. The three currents are represented by the complex values I_a, I_b, and I_c. It is not required that these be balanced, but it is known that

$$I_a + I_b + I_c = 0 \tag{6-55}$$

The voltage drop in wire a can be found by using Eq. 6-27, which gives flux linkages:

$$\begin{aligned}(N\Phi)_a &= \left(I_a \frac{\mu_r}{2} + 2I_a \ln \frac{1}{R} + 2I_b \ln \frac{1}{D} + 2I_c \ln \frac{1}{D}\right)10^{-7} \\ &= \left[I_a\left(\frac{\mu_r}{2} + 2 \ln \frac{1}{R}\right) + (I_b + I_c)2 \ln \frac{1}{D}\right]10^{-7} \\ &= \left[I_a\left(\frac{\mu_r}{2} + 2 \ln \frac{1}{R}\right) - I_a 2 \ln \frac{1}{D}\right]10^{-7} \\ &= I_a\left(\frac{\mu_r}{2} + 2 \ln \frac{D}{R}\right)10^{-7} \end{aligned} \tag{6-56}$$

The inductance of wire a, and hence of any of the three wires, is found by dividing by I_a, giving

$$l = \left(2 \ln \frac{D}{R} + \frac{\mu_r}{2}\right)10^{-7}$$

henrys per meter of wire (6-57)

or

$$l = \left(0.741 \log_{10} \frac{D}{R} + 0.080\ \mu_r\right)10^{-}$$

henrys per mile of wire (6-58)

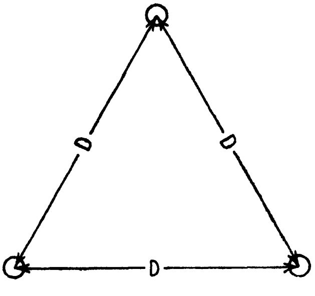

FIG. 6-14. Cross section of a three-phase line, equilateral spacing.

It is a remarkable and valuable coincidence that this formula is identical with the formula (Formula 2 of Table I) for the inductance to neutral of a single-phase line. Let us consider exactly what inductance is given by Eqs. 6-57 and 6-58: it is the inductance that, taken with the current in line a of an equilateral three-phase line, will give the voltage drop along line a—the voltage drop that would be measured on a voltmeter connected as in Fig. 6-5 with the voltmeter leads a very great distance away from the transmission line. Such a voltage drop appears in each wire of a three-phase line, and is quite appropriately considered a voltage *to neutral.*

Next, consider a transmission line in which the three wire-to-wire spacings are not equal, but the wires are *transposed.* One complete transposition "barrel" is shown in Fig. 6-15. Three line positions are indicated as 1, 2, and 3, which identify position on the towers, poles, or crossarms; and the three wires are identified as a, b, and c. Wire a occupies position 1 for one-third of its length, position 2 for one-third of its length, and position 3 for one-third of its length. We now write expressions for flux linkages about conductor a in each of the three sections of the transposition "barrel," multiplying each by $\frac{1}{3}$ because it applies over only one-third of the length, and adding to get total flux linkages per unit length. Equation 6-27 is used:

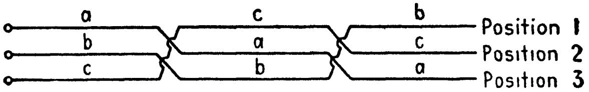

FIG. 6-15. Transposition of three conductors.

$$
\begin{aligned}
(N\Phi)_a &= \frac{1}{3}\left(I_a \frac{\mu_r}{2} + 2I_a \ln \frac{1}{R} + 2I_b \ln \frac{1}{D_{12}} + 2I_c \ln \frac{1}{D_{13}}\right)10^{-7} \\
&\quad + \frac{1}{3}\left(I_a \frac{\mu_r}{a} + 2I_a \ln \frac{1}{R} + 2I_b \ln \frac{1}{D_{23}} + 2I_c \ln \frac{1}{D_{21}}\right)10^{-7} \\
&\quad + \frac{1}{3}\left(I_a \frac{\mu_r}{a} + 2I_a \ln \frac{1}{R} + 2I_b \ln \frac{1}{D_{31}} + 2I_c \ln \frac{1}{D_{23}}\right)10^{-7} \\
&= \left[I_a\left(\frac{\mu_r}{2} + 2 \ln \frac{1}{R}\right) + \frac{1}{3}(I_b + I_c)2 \ln \frac{1}{D_{12}D_{23}D_{31}}\right]10^{-7} \\
&= I_a\left(\frac{\mu_r}{2} + 2 \ln \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{R}\right)10^{-7} \qquad (6\text{-}59)
\end{aligned}
$$

The inductance of wire a of the transposed line is now found by dividing by I_a. At the same time it may be noticed that the numerator of the logarithmic term is the geometric mean of the three wire-to-wire distances, and we write GMD for this mean, giving

$$
l = \left(2 \ln \frac{\text{GMD}}{R} + \frac{\mu_r}{2}\right)10^{-7} \qquad (6\text{-}60)
$$

This formula for inductance of the transposed but nonequilateral line is identical with Eq. 6-57 for the equilateral line, and hence with Formula 2 of Table I, *provided* the geometric mean of the three spacings of the transposed line is used as an *equivalent spacing* in the inductance formula.

By using this equivalent spacing, or geometric mean spacing, it is possible to apply Formula 2 of Table I to any fully transposed three-phase line. This is an exceedingly important fact, for most power transmission lines are of this kind and are treated in this way.

Two further comments may be made. Equation 6-59 for the transposed line applies to a line with any geometrical distribution of its conductors, but a fairly common configuration is to have the three *conductors in the same plane.* Sometimes the plane of the conductors is horizontal; sometimes it is vertical. If the three conductors are in the same plane, each of the outer conductors being at a distance D_{12} from the inner conductor, the equivalent geometric mean spacing for the line is $\sqrt[3]{2}\,D_{12} = 1.26\,D_{12}$.

If desired, Eq. 6-60 can be written with the constant term combined with the logarithmic term. Assuming the conducting material to be nonmagnetic, so that $\mu_r = 1$, the inductance in henrys per meter is

$$l = 2 \ln \frac{\text{GMD}}{\text{GMR}}\, 10^{-7} \tag{6-61}$$

where GMR is the geometric mean radius of the conductor. This is exactly the form of Eq. 6-51.

Finally, something must be said about the *nonequilateral, nontransposed* three-phase line. An equation for the flux linkages about one of the conductors of such a line is readily written:

$$(N\Phi)_a = \left[I_a\left(\frac{\mu_r}{2} + 2 \ln \frac{1}{R}\right) + 2I_b \ln \frac{1}{D_{ab}} + 2I_c \ln \frac{1}{D_{ca}}\right]10^{-7} \tag{6-62}$$

This, however, is as far as the discussion can proceed until something more is known. Only if a relation were known between I_a, I_b, and I_c would it be possible to divide the flux linkages by I_a to obtain an inductance. Indeed, in such a system of conductors, with no other limitations, it is not possible to express voltage drop in a conductor in terms of a single inductance, to be multiplied by the rate of change of a single current. It is necessary, if inductance is to be used, to consider the self-inductance of that conductor and the *mutual inductances* between that conductor and the other two conductors of the line. As another method, voltage drop can be computed from flux linkages without reference to inductance. Fortunately, this problem is rarely encountered in practice.

11. Mutual Inductance. If circuits are close together, a changing current in one will induce a voltage in the other. By definition, mutual inductance is the ratio of induced voltage in one circuit to the rate of change of current in the other:

$$L_{21} = -\frac{e_2}{\dfrac{di_1}{dt}} \tag{6-63}$$

Thus, voltage may be induced in one transmission line by current flowing in another. It is this effect, for example, that is often responsible for inductive interference between a power line and a telephone line, or for *crosstalk* between two telephone lines.

Figure 6-16 shows a two-wire line, *m* and *n*, and a parallel conductor *a* that is carrying current I_a. Flux produced by the current I_a will link the circuit *mn*, but the only flux that need be considered is that between a circle of radius D_{am} and a circle of radius D_{an}. Equation 6-11 gives the flux linkages with the circuit *mn* if the integration is between limits D_{am} and D_{an}:

$$(N\Phi) = \frac{\mu I_a}{2\pi} \ln \frac{D_{an}}{D_{am}} \tag{6-64}$$

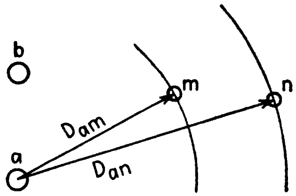

FIG. 6-16. Cross section of two lines with mutual inductance.

Using the permeability of empty space, $4\pi \times 10^{-7}$, and dividing by current to find inductance, the mutual inductance in henrys per meter is

$$l_{21} = 2 \ln \frac{D_{an}}{D_{am}} 10^{-7} = 4.606 \log_{10} \frac{D_{an}}{D_{am}} 10^{-7} \tag{6-65}$$

It will be recognized that this expression is incomplete, for the current I_a cannot exist unless a return path is provided in one or more additional conductors. Other flux linkages with the circuit *mn* will be produced by the current in these other conductors. There must be as many expressions like Eq. 6-64 as there are conductors carrying current. Thus, if a conductor *b* is added to Fig. 6-16 to complete the circuit, mutual flux linkages will be

$$(N\Phi) = \frac{\mu I_a}{2\pi} \ln \frac{D_{an}}{D_{am}} + \frac{\mu I_b}{2\pi} \ln \frac{D_{bn}}{D_{bm}} \tag{6-66}$$

and since $I_b = -I_a$, the mutual inductance in henrys per meter is

$$l_{21} = 2 \ln \frac{D_{an}D_{bm}}{D_{am}D_{bn}} \tag{6-67}$$

If the line carrying current is a three-phase line, and has conductors *a*, *b*, and *c*, an equation similar to 6-66, but with three terms, is written, and induced voltage can be determined if the currents are known. There is no single mutual inductance.

12. Practical Determination of Inductance. Inductance of two-wire lines and of coaxial lines can be computed from the formulas of Table I. These formulas can be used for equilateral or transposed three-phase lines also, by proper interpretation, as has been explained. For less simple types of lines, Eq. 6-51 is of general application, as is illustrated with multiple conductors, stranded conductors, tubular conductors, and conductors of rectangular cross section. As a practical matter, however, the formulas of Table I are adequate for many purposes.

Values of inductance are tabulated in electrical reference books, and several kinds of tabulation are found. It is possible, for instance, to head the columns of a table with values of conductor radius and to index the rows of the table with values of spacing between wires; entries in the table are then the corresponding values of inductance. Such a table assumes uniform distribution of current in the conductor, and it also assumes that the conductor is a solid round wire or else it assumes some particular design of conductor, and a complete table is required for each variety of shape and stranding. In a variation of this type of table, the entries are not inductance in henrys or millihenrys, but reactance in ohms, at a given frequency that is usually 60 cycles. This is perhaps the most common type of table, but it is the most restricted in application.

Another type of tabulation takes advantage of the form of Eqs. 6-51 and 6-61 to permit separation of the data. Equation 6-51 can be written to give inductance in henrys per mile of wire as

$$\begin{aligned} l &= 0.7411 \log \frac{\text{GMD}}{\text{GMR}}\, 10^{-3} \\ &= 0.7411\left(\log \frac{1}{\text{GMR}}\right)10^{-3} + 0.7411(\log \text{GMD})10^{-3} \qquad (6\text{-}68) \end{aligned}$$

The first term of this expression relates to the conductor, to its size, shape, stranding, permeability, and its current distribution if for any reason a nonuniform distribution is to be taken into account. The second term, on the contrary, has nothing to do with the nature of the conductors, and is determined solely by the line spacing. It is convenient, then, to express inductance as the sum of two terms, the first being determined by the design of the conductor and the second being determined by the distance between lines, and to tabulate these quantities separately.

Calling the first term, the *conductor component*, l_a, and the second term, the *spacing component*, l_d, we write

$$l = l_a + l_d \qquad (6\text{-}69)$$

A compact table can then be prepared for l_d in terms of the line spacing (for a three-phase line, the geometric mean of the spacings) of which the following Table 6-4 is an abridged example. Another brief table gives l_a for a particular type of conductor. The following Table 6-3 is an example for solid wire of copper or other nonmagnetic material. Entries in the table are for standard copper wire sizes, except the last entry which is a common size of telephone wire. As an example of use of these tables, the inductance of a pair of No. 6 wires spaced 2 feet apart is 1.689 + 0.223 = 1.91 millihenrys per mile of wire or 3.82 millihenrys per loop mile.

If inductance is required for some other type of conductor, such as seven-strand twisted copper cable, a new table similar to Table 6-3 is needed, with one entry for each size and pattern of stranding. Table 6-4 remains the same for any conductor.

Note that if the spacing is less than 1 foot, the value of l_d, the spacing component, is negative. This results from the use of feet as units in Eq. 6-68; any other units of length could have been used equally well, provided the same unit was used in computing both l_a and l_d. Table 6-3 is sometimes considered to give the inductance owing to flux that lies within a 1-foot radius, and Table 6-4 the inductance owing to flux beyond a 1-foot radius; this concept can be used if it is found to be helpful, but it is not necessary and is likely to raise questions.

TABLE 6-3

INDUCTANCE:
CONDUCTOR COMPONENT

AWG or B&S gauge No.	Outside diameter, inches	l_a, henrys to neutral per mile of wire
2	0.258	1.540×10^{-3}
3	0.229	1.577×10^{-3}
4	0.204	1.615×10^{-3}
5	0.182	1.652×10^{-3}
6	0.162	1.689×10^{-3}
7	0.144	1.726×10^{-3}
8	0.128	1.764×10^{-3}
9	0.114	1.801×10^{-3}
10	0.102	1.838×10^{-3}
—	0.104	1.832×10^{-3}

TABLE 6-4

INDUCTANCE:
SPACING COMPONENT

Separation of conductors GMD, feet	l_d, henrys to neutral per mile of wire
1	0
2	0.223×10^{-3}
3	0.354×10^{-3}
4	0.446×10^{-3}
5	0.518×10^{-3}
6	0.577×10^{-3}
7	0.626×10^{-3}
8	0.669×10^{-3}
9	0.707×10^{-3}
10	0.741×10^{-3}

Since tables of inductance are most often used for power lines, they commonly give values of 60-cycle reactance[10] rather than inductance. The entries in such tables are x_a and x_d, which are equal to the respective inductance terms multiplied by $2\pi f$ or, at 60 cycles, approximately 377. The total reactance, $x = x_a + x_d$, is the reactance per mile to neutral. It may be emphasized again that the phrase *to neutral* has reference to voltage: when this reactance is multiplied by line current it gives the voltage drop *to neutral* (per unit length of conductor).

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[10] Examples of such tables are given in Appendix 3.

Three-phase Spacing. It has been shown that the correct value of separation to use in finding the *spacing component* of inductance or reactance of a transposed three-phase line is the geometric mean of the three line-to-line spacings (the cube root of the product of the three spacings). An alternative method that is sometimes simpler in actual computation is to read from the table a spacing component corresponding to each of the three line-to-line spacings; the arithmetic mean of these three is the actual spacing component to apply to each phase of the line. This is true because the spacing component, as in Eq. 6-68, is a logarithmic function of spacing, and the sum of the logarithms is equivalent to the product of the spacings.

13. Capacitance. Capacitance is a measure of the reactive charging current to a transmission line. When voltage is applied between line conductors, and an electric field is established in the space between the conductors, electric charge will be located on the surface of the conductors as it is on the plates of a charged condenser.

Consider a long, isolated conductor; it has circular cross section as in Fig. 6-17. Because of symmetry, the electric field about it is radial. The electric field strength at the surface of the conductor is equal to the charge density on the conductor surface divided by the dielectric constant of the space. Elsewhere outside the conductor, the field strength is inversely proportional to the radius. Thus, at any point outside the conductor, the electric field strength in volts per meter is

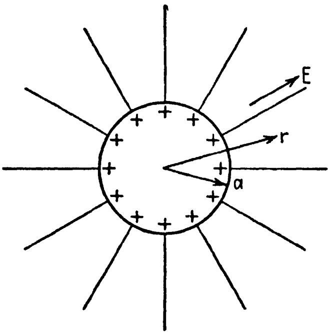

FIG. 6-17. Electric field about a conductor.

$$E = \frac{Q}{2\pi\epsilon r} \tag{6-70}$$

where Q is charge on unit length of the conductor (coulombs per meter)
ϵ is the dielectric constant of the region around the conductor, 8.855×10^{-12} in empty space or, approximately, in air, (farads per meter)
r is radial distance from the center of the conductor (meters)

Potential is the integral of electric field strength. Hence, the potential difference between the surface of the conductor, at radius a, and any point at a greater radial distance b, is

$$V = \int_a^b E \cdot dr = \int_a^b \frac{Q}{2\pi\epsilon r}\,dr = \frac{Q}{2\pi\epsilon}\ln\frac{b}{a} \tag{6-71}$$

The *potential difference* between one body and another can be found, using the principle of superposition, as the sum of the potential difference owing to charge on one of those bodies alone, plus the potential difference owing to charge on the other of those bodies alone, plus the potential difference between those bodies (when uncharged) owing to charge on any other body or bodies in the neighborhood.

Capacitance between two isolated bodies, such as the two wires of a transmission line, is the ratio of the magnitude of the charge on either one of them to the potential difference between them:

$$C = \frac{Q}{V} \tag{6-72}$$

If there are more than two charged bodies present, *coefficients of induction* (or *mutual capacitance*) are required,[11] in general, but various artifices make them unnecessary in most transmission-line problems.

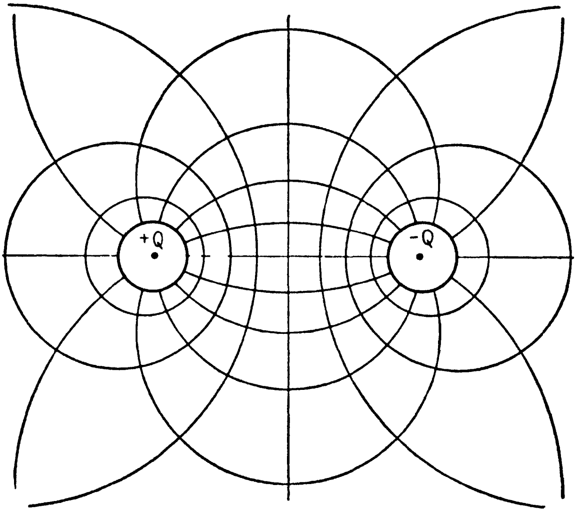

FIG. 6-18. Fields between two parallel conductors.

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[11] See J. C. Maxwell, "A Treatise on Electricity and Magnetism," Sec. 87 (Ref. 17); or S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio," Sec. 6.24 (Ref. 23), or, preferably, both. Any fairly advanced electrical physics book will serve.

Capacitance and inductance are considered in this chapter with reference to nonchanging electric and magnetic fields. As mentioned in Chap. 1, values so obtained are correct for transmission lines at all frequencies, provided line resistance

Two-wire Line. Let us consider the capacitance of a line of two parallel wires, such as that of Fig. 6-18. There is an electric charge of Q on one wire, and $-Q$ on the other, per meter of length. Let us assume that the electric field produced by one of these wires is as given by Eq. 6-70. This is an approximation, for the equation was written for an isolated wire; in making this approximation we are assuming that the wires are small enough and far enough apart that the mere presence of the metal of the second wire does not appreciably alter the field that is due to the charge on the first wire alone.[12]

Potential difference between the two wires is to be found. With the assumption that Eq. 6-70 gives the electric field, Eq. 6-71 gives the potential difference owing to charge on one of the wires alone. Charge on the other wire contributes equally to the potential difference. The charges are equal in magnitude; they are opposite in sign, but the direction of integration is reversed, and the total potential difference is twice that produced by the charge on one wire:

$$V_{(\text{wire to wire})} = \frac{2Q}{2\pi\epsilon} \ln \frac{b}{a} \tag{6-73}$$

Since capacitance is Q/V by definition, as in Eq. 6-72, we find the capacitance per unit length of the transmission line to be

$$c = \frac{\pi\epsilon}{\ln \dfrac{b}{a}} = \frac{12.08 \times 10^{-12}}{\log_{10} \dfrac{b}{a}} \tag{6-74}$$

and this is Formula 6 of Table I. This is the capacitance of the two-wire line in farads per meter.

14. Computation from Velocity. It is evident that there are striking similarities between the computation of capacitance and the computation of inductance. These similarities are inherent in the nature of the electric

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is not excessive. The classic treatment of transmission lines, including the differential equations of Chap. 1, is valid if the axial component of electric field in the dielectric (rI at the conductor surface) is small compared with the transverse electric field, and the transverse electric field in the conductor is small compared with the axial field (the latter condition is satisfied by any ordinary metal). This is discussed by Ramo and Whinnery, Sec. 8.10 (Ref. 23).

[12] This approximation can be avoided as follows. Consider two parallel charged filaments; mere line charges. The electric field about each is accurately given by Eq. 6-70. The total electric field between them is as shown in Fig. 6-18; the filaments, represented by the two dots, are connected by lines of electric force and are surrounded by families of equipotential cylindrical surfaces (not coaxial). Any one of these equipotential surfaces can be the surface of an actual conductor without altering the electric field external to such a conductor. This gives an exact

and magnetic fields, for the magnitudes of both fields are inversely proportional to radius and hence integrate to logarithmic functions. Inductance has a logarithm in the numerator, and capacitance has the same logarithm in the denominator.

This leads to the following significant relation. If the capacitance of a line is multiplied by the external inductance of the line, the product is $\mu\epsilon$. This is a constant: for any line in air, the product of capacitance in farads per meter and external inductance in henrys per meter is 0.111×10^{-16}. A means for computing capacitance from known inductance is now apparent.

By *external* inductance is meant that part of the inductance that results from magnetic flux external to the transmission-line conductors. Those magnetic flux lines of small radius that circulate within the conductors are not part of the external inductance; they constitute the internal inductance. In Eq. 6-17, for example, $(\mu/\pi) \ln b/a$ is the external inductance of a two-wire line, and $\mu/4\pi$ is the internal inductance.

To demonstrate the above relation, consider the two-wire line. External inductance from Eq. 6-17 and capacitance from Eq. 6-74 are multiplied:

$$l_e c = \left(\frac{\mu}{\pi} \ln \frac{b}{a}\right) \frac{\pi\epsilon}{\ln \dfrac{b}{a}} = \mu\epsilon \tag{6-75}$$

This simple result is true for any line, and not only for the two-wire line. A general demonstration comes from electromagnetic-wave theory, and is beyond the scope of this book, but the basic principle is as follows.

The velocity of propagation on a line without loss, and without magnetic field within the conductors, is, by Eq. 3-8, $1/\sqrt{l_e c}$. It can be shown that the velocity of any such transverse electromagnetic wave, regardless of the shape, size, or arrangement of the conductors, is $1/\sqrt{\mu\epsilon}$. (Because light is a transverse electromagnetic wave, this is also the velocity of light in the same medium.) Hence $l_e c = \mu\epsilon$.

It follows from this relation that the velocity is the square root of the

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formulation of the electric field between two parallel circular cylindrical conductors. The capacitance can be found. (See any book on electricity and magnetism.) The result is

$$c = \frac{\pi\epsilon}{\ln\left[\dfrac{b}{2a} + \sqrt{\left(\dfrac{b}{2a}\right)^2 - 1}\right]} \tag{6-74a}$$

Equation 6-74 is obviously a good approximation for Eq. 6-74*a* if b/a is large; in fact, the error is less than 1 per cent if b/a is greater than 7.2. For most two-wire lines, b/a is several hundred, and the approximation is quite satisfactory.

reciprocal of the constant in Eq. 6-75. The value given above for the constant, 0.111×10^{-16}, is computed from the numerical values of Eqs. 6-17*a* and 6-74, and applies in empty space or in air; the velocity in empty space or in air is therefore 3×10^8 meters per second.[13] A similar computation in Eq. 3-10 gives the speed of wave propagation in miles per second.

The immediate practical value of Eq. 6-75 is that it offers an easy means of computing the capacitance of any transmission line if inductance is known. Many pages of this chapter have been devoted to the computation of inductance of lines of various configurations. It is now sufficient to say that capacitance can be found by dividing the constant of Eq. 6-75 by the external inductance of the line. This is true for coaxial lines, multiconductor lines of any arrangement, and polyphase lines, as well as for the simple two-wire line.

It is essential in computing capacitance by this means to use only the *external* inductance of the line. Two forms of inductance formulas have been given. In Formulas 1, 2, 3, and 7 of Table I, the internal inductance can be eliminated by making $K_\varphi = 0$. The remaining term of the formula, which is logarithmic, gives the correct value of external inductance, as explained above.

Equation 6-51 is an expression for inductance that uses the geometric mean radius of a conductor or group of conductors. To avoid the internal inductance, and obtain only the external inductance, when the expression is in this form, assume that all the current is at the surface of the conductors; in other words, solid conductors must be replaced by tubes of the same size and shape with infinitely thin walls. Hence, to determine the *external* inductance of a conductor of circular cross section, use the *actual* radius in place of the geometric mean radius in this formula and in others like it.

Having found the external inductance l_e in henrys per meter, the capacitance c in farads per meter is

$$c = \frac{0.1113 \times 10^{-16}}{l_e} \epsilon_r \mu_r \tag{6-76}$$

where ϵ_r is the relative dielectric constant of the space between conductors ($\epsilon_r = 1$ for empty space or air)

μ_r is the relative permeability of the space between conductors ($\mu_r = 1$ for air or any substance that is not ferromagnetic)

15. Three-phase Capacitance. Three-phase capacitance to neutral can be computed by writing equations for the potential of each of the conductors resulting from charges on that conductor and each of the others, and solving simultaneously, with the provision that the charges add to zero. However, since the inductance of an equilateral or transposed three-phase line has already been determined, we can deduce the capacitance from the inductance and the known velocity of propagation. Formula 2 of Table I gives the inductance to neutral of a three-phase line, and it follows that Formula 5 gives the capacitance to neutral of a three-phase line, for the two formulas are related by Eq. 6-76 (with due regard for measurement in miles instead of meters).

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[13] More precisely, the speed of light in free space is 2.99789×10^8; see Kees Bol, A Determination of the Speed of Light by the Resonant Cavity Method, *Phys. Rev.*, Vol. 80, p. 298, October 15, 1950.

Formula 5 gives the capacitance to neutral of an equilateral three-phase line. It is also applicable to a nonequilateral three-phase line if the three conductors are fully transposed and the geometric mean of the three actual line-to-line spacings is used as the "spacing between wire centers" in the formula.

Tables. Tables of capacitance as a function of conductor size and spacing are found in various reference books. For use with power lines, either single-phase or three-phase, it is convenient to use a tabular form similar to that discussed in Sec. 12 for inductance. Formula 5 of Table I is used to write an equation for the reciprocal of the 60-cycle susceptance to neutral per mile of wire.

Following W. A. Lewis (Ref. 16), to whom this method is due, we shall call this value the capacitive reactance to neutral per mile of wire, and denote it by x', writing

$$\begin{aligned} x' &= -\frac{1}{2\pi fc} \\ &= \frac{-10^9}{2\pi f(38.88)} \log_{10} \frac{b}{a} \\ &= \frac{-4.09 \times 10^6}{f}\left(\log \frac{1}{a} + \log b\right) \qquad (6\text{-}77) \end{aligned}$$

Specifically at a 60-cycle frequency,

$$x' = -0.0682 \times 10^6\left(\log \frac{1}{a} + \log b\right) \qquad (6\text{-}78)$$

The first logarithmic term in Eq. 6-78 relates to conductor size only; the second relates to spacing between conductors only. We may therefore do as we did with inductance, dividing the total value for x' into a *conductor component* x_a', and a *spacing component* x_d', and writing

$$x' = x_a' + x_d' \qquad (6\text{-}79)$$

Tables of x_a' are then prepared for conductors of various kinds and sizes, taking stranding and other pertinent factors into account, and a separate table is computed for x_d' that is concerned only with separation between conductors and that is equally applicable to all types of conductors. Such tables are given in Appendix 3; more complete tables are available in various reference books (particularly the Westinghouse "Reference Book," Ref. 18).

To use such tables, find x_a' and x_d' from the tables, add, and take the reciprocal; this gives the capacitive susceptance per mile of line. Then multiply by the length of the line if susceptance of the total line is required. (It will be noted that x' must be *divided* by the length of the line if the capacitive reactance of the entire line is to be found.)

Let us consider carefully what is meant by *capacitance to neutral* of a three-phase line. It is the ratio of charge on one conductor of the line (per unit length) to the voltage from that conductor to neutral. It is not, and has no relation to, any capacitance from line to ground. Neither is it the capacitance to an actual neutral wire. It exists even if there is no neutral wire, and if no ground surface is considered; indeed its value is changed only slightly if an actual neutral wire is brought into the problem, or if the effect of the ground below the transmission line is taken into account.

Charging current to unit length of each conductor of a transmission line is the capacitance per unit length multiplied by the rate of change of the voltage to neutral. For a sinusoidal voltage of frequency f, the charging current is equal to the line capacitance multiplied by $j2\pi f$ times the voltage to neutral. Note that the *capacitance to neutral* is always used with the *voltage to neutral.*

It remains to define the meaning of *voltage to neutral.* It is the potential difference between a conductor of a transmission line and an infinitely distant point.

A three-phase system has three voltages from line to neutral. If the three-phase voltages are balanced, the three voltages to neutral are equal in magnitude and 120 degrees apart in phase. The sum of the three line-to-neutral voltages in a balanced system is zero. This familiar case is well known, and needs no explanation. It is recognized that these voltages to neutral are just as useful for purposes of analysis on a three-wire line, without a neutral wire, as they are if a physical neutral actually exists.

It is not necessary, however, for the present treatment of line capacitance to be limited to a line with balanced voltages. The only limitation that has been imposed is that there shall be only three conductors and that current does not flow in a fourth wire or in ground[14] (this was stipulated in computing inductance). As there are only three conductors, the sum of the charges on the three conductors must equal zero. Consequently, the sum of the three line-to-neutral voltages must be zero.

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[14] In terms of symmetrical components, the present discussion is limited to positive- and negative-sequence systems; zero-sequence capacitance, which depends on the capacitance from the three wires of the line to a physical neutral or to the

The voltages from line to neutral combine in pairs to give the line-to-line voltages. From this, and the fact that they must add to zero, they can be computed as

$$V_{an} = \tfrac{1}{3}(V_{ab} - V_{ca}) \tag{6-80}$$

$$V_{bn} = \tfrac{1}{3}(V_{bc} - V_{ab}) \tag{6-81}$$

$$V_{cn} = \tfrac{1}{3}(V_{ca} - V_{bc}) \tag{6-82}$$

The mere *presence of a ground surface* below a three-phase line affects the capacitance slightly, even though the sum of the line charges is zero. The ground behaves simply as an uncharged conducting body. A certain electric field surrounds the three-wire transmission line in the absence of ground; if the surface of the earth is then brought near the line, the electric field is altered and the capacitance is correspondingly changed. However, the effect is so small with three-phase transmission lines of ordinary dimensions that it can be entirely neglected.

The *mutual capacitance* between two three-phase lines that run near each other, perhaps even being suspended from the same steel towers, is a factor that must not be forgotten. For lines of usual proportions, this factor, also, has negligible affect on the three-phase capacitance of the lines.

Mutual capacitance is of great importance, however, when two lines run side by side and the electric field from one line to ground induces potential from line to ground in the other. Similarly mutual inductance between two such lines is important if the current flowing in the three conductors of one line, and returning in ground, produces a magnetic field that links with the other line and induces potential from that line to ground. (These are *zero-sequence* current and voltage; see footnote 9). However, we shall continue to exclude consideration of current or charge in the ground, and shall not discuss these phenomena.

ground surface, is a different quantity. Equations for zero-sequence resistance, reactance, and capacitance are given in Appendix 3. Zero-sequence quantities are not considered here but may be found in books relating to symmetrical components, such as Refs. 18 and 36. The capacitance that is here called *capacitance to neutral* relates positive-sequence charging current to positive-sequence voltage, negative-sequence charging current to negative-sequence voltage, and hence any line charging current to the corresponding line-to-neutral voltage, provided the line-to-neutral voltage is defined, as in this section, to have no zero-sequence component.

PROBLEMS

6-1. Determine from a handbook, or from any other convenient reference, the relative magnetic permeabilities of copper and aluminum. (Relative to permeability of free space.)

6-2. A four-wire line as shown in Fig. 6-7 consists of four No. 10 AWG copper wires at the corners of an 8-inch square. Find the low-frequency inductance of such a line 100 meters long.

6-3. Repeat Prob. 6-2 for a line of No. 8 AWG copper wires at the corners of a 10-inch square.

6-4. The wires of Fig. 6-7 are moved to be at the corners of a rectangle instead of a square; that is, the distance from a_1 to b_1 may differ from the distance from a_1 to b_2. Derive an expression for inductance to replace Eq. 6-31 for such a line.

6-5. All four wires of Fig. 6-7 carry current in the same direction; a fifth wire is added at the center of the square to carry all the return current. Derive an expression for inductance of this five-wire line.

6-6. In Fig. 6-7, wires a_1 and b_2 carry current in one direction, and wires b_1 and a_2 carry return current in the other direction. (It may be convenient to reletter the wires in solving this problem.) Find inductance. Is it higher or lower than the inductance given by Eq. 6-31 for the original scheme of connection?

6-7. A six-wire line has wires arranged at the vertices of a regular hexagon. If these are designated by letters a to f in consecutive order around the hexagon, wires a, c, and e carry current in one direction and b, d, and f constitute the return conductor. Derive an expression for the inductance of the six-wire line.

6-8. The four-wire line of Fig. 6-8 is reconnected so that wires a_1 and b_2 carry current in one direction, and wires a_2 and b_1 carry return current in the other direction. Find inductance. Is it greater or less than with the connection that leads to Eq. 6-40?

6-9. The wires of the four-wire line of Fig. 6-8 are No. 10 AWG copper. The spacing $D = 12$ inches. By what percentage is the resistance loss in the wires (connected as shown in Fig. 6-8) decreased by transposition (as in Fig. 6-9)? Frequency is 1,000 cycles per second.

6-10. Repeat Prob. 6-9 for No. 18 wires with 10-inch spacing.

6-11. How does current divide between the wires of Fig. 6-8 at zero frequency? at high frequency (allowing frequency to become great without limit)? Note that the high-frequency division is independent of both frequency and wire resistance, a point of interest in connection with *proximity effect.*

6-12. Derive Eq. 6-40 from Eqs. 6-36 and 6-37.

6-13. Show whether or not Eq. 6-47 is equivalent to Eq. 6-30.

6-14. Show whether or not Eq. 6-50 is equivalent to Eq. 6-40.

6-15. Use Eq. 6-51 to find inductance of the lines (*a*) of Fig. 6-7; (*b*) of Fig. 6-9; (*c*) of Prob. 6-4; (*d*) of Prob. 6-5; (*e*) of Prob. 6-6; (*f*) of Prob. 6-7; (*g*) of Prob. 6-8.

6-16. Verify the value given in Table 6-1 for the GMR of stranded cable of three strands.

6-17. Find the GMR of a stranded cable of four strands.

6-18. A solid round conductor is 1 inch in diameter. What size conductor of square cross section has equal inductance per unit length?

6-19. A wire table gives resistance per mile of wire. Is this value, or is a multiple of this value, used in finding inpedance: (*a*) per mile of two-wire line; (*b*) to neutral per mile of wire, single phase; (*c*) to neutral, three-phase; (*d*) per loop-mile; (*e*) per mile per phase.

6-20. The transmission line of Fig. 5-3 has copper conductors (General Cable Type H-H) that are essentially tubular. Compute inductance per mile of conductor assuming (*a*) that all current flows at the surface of the conductor, (*b*) that the conductor is a tube, with the outside diameter and copper cross section as given, and (*c*) that the conductor is solid with the same outside diameter.

6-21. The wires *m* and *n* of Fig. 6-16 comprise a single-phase power line; they are 5 feet apart. The wires *a* and *b*, 12 inches apart, are a parallel telephone line. The two lines are on opposite sides of a narrow road, 50 feet apart. Neither line is transposed. Wires *m* and *n* are the same height above ground as *a*. Find voltage induced per mile of telephone line by 100 amperes of 60-cycle current in the power line.

6-22. Repeat Prob. 6-21, assuming wire *b* is in the same plane with *a*, *m*, and *n*, and 12 inches to the left of *a*.

6-23. Use Tables 6-3 and 6-4 to compute inductance of the telephone line of Fig. 5-2.

6-24. A 380-kilovolt, three-phase, 50-cycle transmission line (Swedish State Power Board) uses two steel-aluminum cables, hung close together, to carry current of each phase. The conductors are spaced in a horizontal plane as indicated:

o o o o o o

Each conductor of this "double-conductor" line is similar to 1,033,500-circular-mil ACSR (see Appendix 3). Spacing between phases (center to center) is 12 meters. Assuming transposition, compute and plot inductive reactance to neutral per kilometer of double conductor as a function of the spacing between the two cables of the double conductor (from 10 to 80 centimeters).

Note: The spacing in the accepted design is 45 centimeters, about 0.33 ohms per kilometer. (A. Rusck and B. G. Rathsman, *Elec. Eng.*, Vol. 69, pp. 53–57, January, 1950.)

6-25. Compute capacitance (specify whether it is line-to-line or line-to-neutral) using inductances computed in Probs. 6-2, 6-3, 6-20 (*a*, *b*, and *c*), 6-23, and 6-24.

6-26. Using results of Prob. 6-25, compute the characteristic impedance of each line. (An approximate value using $Z_0 = \sqrt{l/c}$ will be adequate for this problem.) Specify whether Z_0 is line-to-neutral or line-to-line.

CHAPTER 7

LINE CONSTANTS: RESISTANCE, SKIN EFFECT, AND INSULATION LOSS

1. Line Resistance. The resistance of a transmission-line conductor can be computed from the dimensions of the conductor and the resistivity of the material. As a matter of fact, however, such computation does not usually give a precise result, for there are a number of factors that cannot always be determined exactly. The most important uncertainty is often the temperature, which may vary from hour to hour and from place to place along the line, and which is not independent of the line current. Also, the resistivity of the conducting material, even at a given temperature, may be inexactly known, as it depends on the metallurgy and heat-treatment of the material as well as its chemical composition. Thus, annealed copper has not quite the same resistivity as hard-drawn copper; and the proportion of elements in an alloy such as brass will greatly affect the resistivity. Even the exact size of a conductor may be somewhat in doubt; wire, for instance, drawn through a new die is slightly smaller than wire drawn through the same die when it is old and worn, and yet both are nominally the same gauge. Moreover, when there is pronounced skin effect, mere roughness or impurity of the surface may be important. For these various reasons, it is necessary for the engineer, if resistance is important, to consider the probable accuracy of any computed value.

The resistance of a conductor of uniform cross section is

$$R = \rho \frac{l}{A} \tag{7-1}$$

where R is resistance (ohms)
ρ is resistivity (meter-ohms)
l is length in the direction of current (meters)
A is cross-sectional area normal to the current (square meters)

This basic equation can be used with various unit systems (although mks units are here specified), and resistivity values are published in quite a variety of units. Thus the resistivity of standard annealed copper at 20°C. is, alternatively,

1.7241×10^{-8} meter-ohm

1.7241×10^{-6} centimeter-ohm
0.6788×10^{-6} inch-ohm
0.017241 ohm for meter length and 1-square-millimeter cross section
10.371 ohms for foot length and 1-circular-mil cross section

The last figure is convenient for round wires and English units, the *circular mil* being the area of a circle of 0.001-inch diameter. The next to last figure above is the "International Annealed Copper Standard."[1] The first two values are in the common metric units of resistivity.[2]

The variation of resistivity of copper with temperature is commonly expressed by saying that the resistivity is proportional to the temperature above −234.5°C. This can be expressed symbolically as

$$\frac{\rho_2}{\rho_1} = \frac{t_2 + T}{t_1 + T} \tag{7-1a}$$

where ρ_1 and ρ_2 are the resistivities at centigrade temperatures t_2 and t_1, respectively, and T, for annealed copper, is 234.5. This linear relation is quite accurate over a wide range of temperature, but of course fails to apply at temperatures approaching −234.5°C.

Industrial copper is more commonly *hard-drawn* or *medium hard-drawn* than annealed. The classification of hardness that defines the various grades is based on tensile strength, as in Table 7-1 (abstracted from tables of the American Society of Testing Materials Standard).

The metallurgical treatment of the copper affects its conductivity by a few per cent. Typical values are given in Table 7-2, together with data on various other metals. The data for copper and aluminum as given in the table have been carefully worked out by various authorities and standardized; for other metals, the electrical characteristics are slightly variable, depending on metallurgy and purity. In general, any impurity

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[1] See *National Bureau of Standards Circular* 31, or *Transactions of the American Institute of Electrical Engineers*, Vol. 32, p. 2164, 1913. "The following values should be taken as normal for annealed standard copper. At 20 deg. cent., the resistance of an annealed copper wire 1 meter long and having a uniform cross-section of 1 sq. mm. is 1/58 ohm = 0.017241 . . . ohm. At 20 deg. cent., the coefficient of variation of resistance with temperature of annealed copper, measured between potential terminals rigidly attached to the wire (constant mass), is 0.00393 = 1/254.5 per deg. cent. . . . [These] are mean values deduced from a large number of tests."

[2] The incorrect usage of "ohm per cubic centimeter" is still occasionally seen, wrongly used in place of *ohm-centimeter*. It is true that the resistance in ohms, measured between opposite faces of a cube of copper 1 centimeter on each edge, is numerically equal to the resistivity in centimeter-ohms, but it is obvious that resistance is not proportional to the volume of copper in cubic centimeters. Resistivity is obtained by dividing resistance by length and multiplying by area, and this must be indicated in its dimensions.

TABLE 7-1
HARDNESS OF COPPER WIRE

AWG size	Diameter, inches	Tensile strength, pounds per square inch		
		Soft or annealed	Medium hard-drawn	Hard-drawn
0000	0.460	36,000	42,000–49,000	49,000
10	0.102	38,500	50,000–57,330	64,900
18	0.040	38,500	53,000–60,000	67,000

TABLE 7-2
RESISTIVITY OF METALS AND TEMPERATURE COEFFICIENTS OF RESISTIVITY

Metal	Resistivity at 20°C, meter-ohms	Conductivity, per cent of standard annealed copper	Temperature coefficient of resistivity per degree centigrade at 20°C	Reference temperature, T in Eq. 7-1*a*
Copper, standard annealed.	1.7241×10^{-8}	100	0.00393	234.5
Copper, soft-drawn.......	1.76×10^{-8}	98	0.00385	240
Copper, medium hard-drawn.................	$1.76\text{–}1.78 \times 10^{-8}$	97–98	0.00383 av	241
Copper, hard-drawn......	1.77×10^{-8}	97.3	0.00382	242
Aluminum, 99.97% pure..	2.66×10^{-8}	64.6	0.00427	214
Aluminum, average commercial hard-drawn.....	2.83×10^{-8}	61.0	0.00403	228
Brass, typical............	$7\text{–}8 \times 10^{-8}$	21–25	0.002	
Gold....................	2.44×10^{-8}	70.8	0.0034	
Iron, pure...............	10×10^{-8}	17	0.005	
Platinum................	10×10^{-8}	17	0.003	
Silver...................	1.63×10^{-8}	106	0.0038	
Steel, carbon, medium to hard..................	$20\text{–}50 \times 10^{-8}$	3–9	0.002–0.005	
Tin......................	11.5×10^{-8}	15	0.0042	
Tungsten................	5.51×10^{-8}	31	0.0045	
Zinc....................	5.8×10^{-8}	30	0.0037	

increases the resistivity, and in some cases a small amount of impurity has a very large effect on resistivity.

2. Skin Effect. For a-c work, or, in general, for any rapidly changing current, the effective resistance of a wire is dependent on *skin effect.* The true, or "ohmic," resistance of a conductor applies, by definition, when current is uniformly distributed through the cross section of the conductor; that is, when current density is everywhere the same. With alternating current, this is not the case, and there is greater current density near the surface of the conductor. This is known as *skin effect,* for at very high frequency it is so extreme that nearly all the current flows in a thin surface layer of the conducting material. Skin effect has two results: it increases the effective resistance of the conductor, and it slightly decreases the inductance.

The increase in effective resistance is much the more important aspect of skin effect. Effective resistance is that quantity which may be multiplied by the square of the current to give power loss. Any unequal distribution of current density within a conductor increases power loss (the total current remaining the same) for, although loss is decreased in those parts of the conductor in which the current density is lessened, it is increased to a greater extent in other parts of the conductor that have an increased current density. The net increase of loss, and consequent increase of effective resistance, results from the fact that power loss is proportional to the square (and not the first power) of current.

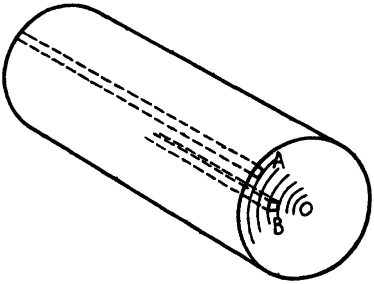

FIG. 7-1. Section of a round wire, showing imaginary filaments.

The basic cause of skin effect is quite simple. Consider a conductor, such as is indicated in Fig. 7-1, of solid copper. We wish to give special attention to certain filaments, such as A and B, which are indicated in the diagram although they are physically indistinguishable from the rest of the copper. The hypothetical filament A is at the surface, and B is nearer the center of the conductor. It is assumed that the conductor is uniform in size and shape, so that every cross section is like every other cross section. It is also assumed that total current is the same at every cross section; that is, all the current entering one end of the conductor leaves at the other. Current must therefore flow parallel to the filaments indicated, which are parallel to the axis of the wire, and any cross-sectional plane must be an equipotential surface.

This last fact is the basis for determining current distribution. All filaments, such as A and B, must have the same voltage drop; cross-sectional planes are equipotential surfaces, and this requires that the impedance drop of voltage must be the same in every filament in the length between any one cross-sectional plane and another. With direct current, this condition can be satisfied only by uniform current density, giving uniform resistance drop. But when the current is alternating (or otherwise changing with time) the resistance drop will not be the only voltage to be taken into account. There will also be an induced voltage in each filament resulting from the changing magnetic field that is caused by current in the conductor itself. This is an inductive reactance drop. Then the resistance drop plus the reactance drop in one filament must be equal to the resistance drop plus the reactance drop in every other filament. It will be recalled that magnetic flux lines circle about the axis of the conductor. Some will enclose B without enclosing A. The reactance of a filament near the surface (at A) is therefore less than the reactance of a filament nearer the center (at B), because filament A is linked by fewer lines of magnetic flux. In order to produce equal voltage drops, current density must be greater nearer the surface to compensate for the lesser reactance. This is skin effect.

Quantitatively, we may write an equation for the voltage drop along any filament, and this drop must be constant. Actually, it is more convenient to write the difference between the drop along one filament and that along another filament, and say this difference must be zero. The solution of such an equation, which is a differential equation, then gives the distribution of current density that exists within the conductor.

One or two preliminary steps are necessary before starting on this solution. Let us use I to mean total current in the conductor; in the following discussion of skin effect, I is the instantaneous value of total current. The symbol i will be used for current density (amperes per square meter), and the relation between current and current density is

$$I = \int i \cdot dA \tag{7-2}$$

(Both I and i have different meanings in other parts of this book, and the meanings here given are used only in the discussion of skin effect.) Equation 7-2 says that current density is integrated over the cross-sectional area of a conductor to find total current in the conductor.

Next, a modified form of Ohm's law is wanted. If the current density in a conductor is uniform, with a value i, the resistivity being ρ, the cross-sectional area of the conductor A, and the length of the conductor s, the voltage drop in the conductor is

$$RI = \rho \frac{s}{A} \cdot iA = \rho si \tag{7-3}$$

This gives a means of finding the ohmic voltage drop along one of the filaments of Fig. 7-1 when the current density and resistivity are known.

Another view of the same conductor is shown in Fig. 7-2. We are interested in two filaments, AB and $A'B'$, of length s. AB is at radius r, and $A'B'$ at $r + dr$. If the current density along AB is i, the current density along $A'B'$ is $i + \frac{\partial i}{\partial r}\,dr$. Assuming uniform resistivity of the conductor material, the ohmic voltage drop along AB is $\rho s i$, and the ohmic drop along $A'B'$ is $\rho s\left(i + \frac{\partial i}{\partial r}\,dr\right)$. The difference between these two values, which may also be thought of as the summation of voltage drop around the rectangular path $A'B'BA$, is

$$\rho s \frac{\partial i}{\partial r}\,dr \tag{7-4}$$

Now consider the reactive effect. Referring again to Fig. 7-2, a certain amount of magnetic flux will pass through the rectangle $A'B'BA$. If we call this flux Φ, the electromotive force induced in the rectangular path $A'B'BA$ is

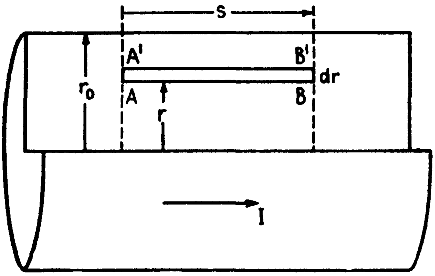

FIG. 7-2. Side view of a round wire.

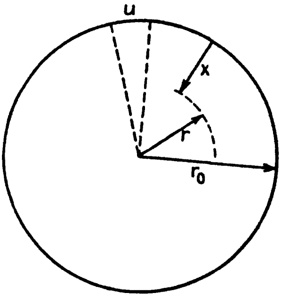

FIG. 7-3. End view of a round wire.

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t} B\cdot(\text{area}) = \frac{\partial B}{\partial t}\, s\cdot dr \tag{7-5}$$

This induced electromotive force is the cause of the inequality of ohmic voltage drop in the filaments AB and $A'B'$ and the electromotive force from Eq. 7-5 can be equated to the difference of ohmic drop from Eq. 7-4:

$$\rho s \frac{\partial i}{\partial r}\,dr = \frac{\partial B}{\partial t}\, s\, dr \tag{7-6}$$

From Eq. 7-6, knowing that $B = \mu H$, we have

$$\frac{\partial H}{\partial t} = \frac{\rho}{\mu}\frac{\partial i}{\partial r} \tag{7-7}$$

which is a differential equation that relates the current distribution in a conductor to the rate of change of magnetic field strength. It may be noted that differentiation of Eq. 7-7 gives

$$\frac{\partial^2 H}{\partial t\, \partial r} = \frac{\rho}{\mu} \frac{\partial^2 i}{\partial r^2} \tag{7-8}$$

This second derivative will be useful a little later.

Equation 7-7 provides that all filaments of a conductor shall have the same voltage between cross-sectional planes. It cannot be solved alone; another equation is needed. The distribution of magnetic field produced by current in a round wire must be taken into account, as follows.

3. Skin Effect in a Round Wire. It was shown in the previous chapter (Eq. 6-4) that the magnetic field of current in a round wire is given by

$$2\pi r H = I \tag{7-9}$$

where H is the magnetic intensity at radius r and I is the total current enclosed within a circle of radius r (see Fig. 7-3). In the previous chapter, it was then assumed that current density was uniform and an expression for magnetic field distribution (Eq. 6-7) was derived. In studying skin effect, such an assumption is, of course, inadmissible, and Eq. 7-9 must be used together with Eq. 7-7 to find the actual distribution of both magnetic field and current density.

To put Eq. 7-9 into a useful form, first use Eq. 7-2 to represent the current within radius r:

$$2\pi r H = \int_0^r i \cdot dA = \int_0^r i \cdot 2\pi r\, dr \tag{7-10}$$

Dividing by 2π and differentiating,

$$\frac{\partial}{\partial r}(rH) = ir \tag{7-11}$$

or

$$i = \frac{H}{r} + \frac{\partial H}{\partial r} \tag{7-12}$$

This gives current density in terms of magnetic intensity. For simultaneous solution with Eq. 7-7 it will be helpful to differentiate again, with respect to time:

$$\frac{\partial i}{\partial t} = \frac{1}{r} \frac{\partial H}{\partial t} + \frac{\partial^2 H}{\partial r\, \partial t} \tag{7-13}$$

It is now possible to eliminate H by substituting Eqs. 7-7 and 7-8 into Eq. 7-13:

$$\frac{\partial i}{\partial t} = \frac{\rho}{\mu}\left(\frac{1}{r}\frac{\partial i}{\partial r} + \frac{\partial^2 i}{\partial r^2}\right) \tag{7-14}$$

This equation has only one dependent variable, and a solution for current density as a function of radius should be possible if its variation with time were known.

Let us limit our discussion to alternating current of a single frequency Current is then conveniently expressed as the real part of an exponential function:

$$I = I_{\max}e^{j\omega t} \tag{7-15}$$

A similar expression must apply to current density:

$$i = i_{\max}e^{j\omega t} \tag{7-16}$$

Differentiating,

$$\frac{\partial i}{\partial t} = j\omega i_{\max}e^{j\omega t} = j\omega i \tag{7-17}$$

This permits elimination of the time derivative from Eq. 7-14, giving

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r}\frac{\partial i}{\partial r} - \frac{j\omega\mu}{\rho} i = 0 \tag{7-18}$$

We now have a differential equation containing only i and r, and this equation will give the distribution of current density in the round conductor as a function of radial distance. Let us postpone actual solution of this equation until later.

4. Skin Effect at a Plane Surface. Equation 7-18, for a round wire, is not difficult to derive, but its solution may not be readily apparent. A similar equation will now be derived for skin effect in a semi-infinite conductor, for in this case the solution will give no difficulty.

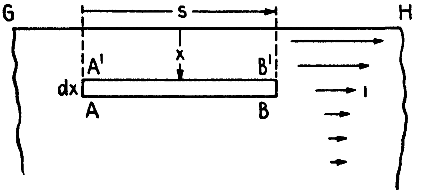

FIG. 7-4. Side view of an infinite conducting block.

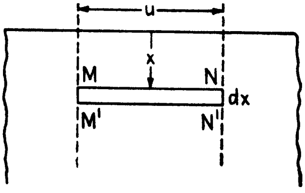

FIG. 7-5. End view of infinite block.

In Fig. 7-4, GH is a portion of the upper surface of a large block of conducting material. Current flows in this material, with density i. There is no current in the space above the surface. The block is so large that it fills all space below the surface GH. Current density is uniform all over the surface but, because of skin effect, current density is less at increasingly greater depths below the surface. The problem is to find current density as a function of depth.

It is convenient to think of this problem as a special case of the problem of the round wire in which the wire is so large, or the skin effect is so marked, that practically all of the current flows quite close to the surface. Figure 7-4 is then a greatly enlarged view, as seen through a microscope, of a small part of Fig. 7-2 near the surface of the conductor. In this enlarged view, GH appears to be a plane surface because the radius of curvature of the surface is great compared to the other dimensions, and particularly compared to the depth of penetration of the current. Figure 7-5 is a diagram looking at the same semi-infinite block in the direction of current flow; it is analogous to Fig. 7-3.

It is no longer feasible to specify current density in terms of radius r. Let us use, instead, the depth x measured into the material from the surface. (As indicated in Fig. 7-3, $x = r_0 - r$.)

The solution then proceeds as for the round wire. The difference between ohmic resistance drops along filaments $A'B'$ and AB of Fig. 7-4 is

$$-\rho s \frac{\partial i}{\partial x}\,dx \tag{7-19}$$

Voltage induced by magnetic flux through the rectangle $A'B'BA$ is (as in Eq. 7-5)

$$\frac{\partial B}{\partial t}\,s\,dx \tag{7-20}$$

and equating these expressions to make the summation of resistance drops around the rectangle $A'B'BA$ equal the electromotive force,

$$\frac{\partial B}{\partial t} = -\rho \frac{\partial i}{\partial x} \tag{7-21}$$

from which

$$\frac{\partial H}{\partial t} = -\frac{\rho}{\mu}\frac{\partial i}{\partial x} \tag{7-22}$$

It is now necessary to find an expression to relate magnetic intensity and current density in a semi-infinite conductor. For a circular wire, Eq. 7-12 gives

$$i = \frac{H}{r} + \frac{\partial H}{\partial r} \tag{7-23}$$

but, for the block, measurement must be in terms of depth x rather than radius r. From the above definition of x,

$$r = r_0 - x \tag{7-24}$$

and

$$\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial r} \tag{7-25}$$

With these relations, Eq. 7-23 becomes

$$i = \frac{H}{r_0 - x} - \frac{\partial H}{\partial x} \tag{7-26}$$

This applies to a circular wire. The semi-infinite block is the limiting case in which r_0 is infinite; when r_0 is infinite, $H/(r_0 - x)$ is zero for finite values of x. Hence, for a semi-infinite block,

$$i = -\frac{\partial H}{\partial x} \tag{7-27}$$

This is the equation needed.

We now differentiate Eq. 7-22 with respect to x, and Eq. 7-27 with respect to t and, by equating the results, H is eliminated:

$$\frac{\partial^2 H}{\partial t\, \partial x} = -\frac{\rho}{\mu}\frac{\partial^2 i}{\partial x^2} = -\frac{\partial i}{\partial t} \tag{7-28}$$

Assuming an alternating current of a single frequency, we use $j\omega i$ (as in Eq. 7-17) for $\partial i/\partial t$, and write

$$\frac{\partial^2 i}{\partial x^2} - \frac{j\omega\mu}{\rho} i = 0 \tag{7-29}$$

This is as far as the solution for skin effect in a circular conductor was carried. A solution for Eq. 7-18 was not obtained. Equation 7-29, however, is of familiar appearance (see Eq. 1-23), and we suspect that it has a solution expressible in exponential form.

Assume as a form of solution of Eq. 7-29

$$i = Ae^{\gamma x} \tag{7-30}$$

A and γ are not determined and, to find what they must be, we differentiate Eq. 7-30 and substitute into Eq. 7-29:

$$\gamma^2 Ae^{\gamma x} - \frac{j\omega\mu}{\rho} Ae^{\gamma x} = 0 \tag{7-31}$$

from which

$$\gamma = \pm\sqrt{\frac{j\omega\mu}{\rho}} = \pm\sqrt{\frac{\omega\mu}{2\rho}}(1 + j) \tag{7-32}$$

This tells us that Eq. 7-30 will be a correct solution of 7-29, provided γ has either the positive or negative value of Eq. 7-32 or, since the equation is linear, the most general solution will be the sum of the two special solutions

$$i = A_1 e^{+\sqrt{\omega\mu/2\rho}(1+j)x} + A_2 e^{-\sqrt{\omega\mu/2\rho}(1+j)x} \tag{7-33}$$

A_1 and A_2 remain to be evaluated.

A_1 can be disposed of easily. Since the exponential to which it is attached must increase without limit as x increases, this term would describe a component of current density becoming infinite at great depths within the conductor. Since infinite current is physically impossible, it is necessary that $A_1 = 0$. This leaves, then, only the second exponential term. To evaluate A_2, consider that if $x = 0$, Eq. 7-33 must give the current density at the surface of the semi-infinite conductor. If we call the current density at the surface i_0, we may write $A_2 = i_0$, and the expression for current density becomes

$$i = i_0 e^{-\sqrt{\omega\mu/2\rho}(1+j)x} = i_0 e^{-\sqrt{\omega\mu/2\rho}\,x} e^{-j\sqrt{\omega\mu/2\rho}\,x} \tag{7-34}$$

5. Current Distribution with Skin Effect. The final form of this equation is particularly interesting. It gives the current density at any depth x in terms of the current density at the surface. The first exponential, being real, shows how rapidly the magnitude of the current density decreases within the conducting material, while the second exponential, with imaginary exponent, gives the phase relation between current density at any depth and that at the surface.

At first thought (or, perhaps, without thought) one might picture current density, as influenced by skin effect, diminishing in the interior of a conductor, but having more or less the same phase at all depths. This is not true. Instead, Eq. 7-34 suggests a concept of current flowing first at the surface and diffusing inward. If current density were visible, it would appear to diffuse into the conductor.

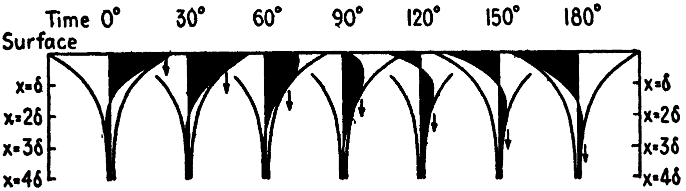

FIG. 7-6. Diffusion of current density.

Figure 7-6 illustrates this concept. The figure shows the distribution of current density within the conductor at six instants of time, the six instants being spaced $\frac{1}{12}$ cycle apart in time, so that the six diagrams provide a moving picture of current density during half a cycle. In the first view (marked 0°), current density at the surface is at its maximum value, directed toward the right. In the second view (marked 30°), current density at the surface is less, while at a depth slightly below the surface, current density has just attained its maximum value. In the third view (marked 60°), current density has reached its maximum at a greater depth. A small arrow marks the depth at which current has reached its maximum value in each of the diagrams, and may be interpreted as marking the crest of a current-density wave that is traveling into the conductor. The phase velocity of such a wave can be computed from the imaginary exponent (analogous to β) of Eq. 7-34, and the rate of attenuation from the real exponent (analogous to α).[3]

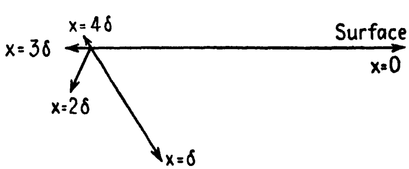

FIG 7-7. Current density at various depths.

Figure 7-7 is another graphic representation of the same information. Several vectors are drawn to represent current density at different depths within the conducting material, relative to current density at the surface. It is convenient to speak of depth in terms of a distance δ which is

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}} \tag{7-35}$$

where ρ is resistivity (meter-ohms); see Table 7-2
ω is $2\pi f$ (radians per second)
μ is absolute permeability (henrys per meter); 1.257×10^{-6} if nonmagnetic
δ is measured in meters

This is convenient because at the depth $x = \delta$ the exponents of Eq. 7-34 become unity. At $x = \delta$, the magnitude of the current-density vector is

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[3] This is properly diffusion rather than wave propagation, and α and β are not independent. Equation 7-28, which may be written

$$\frac{\partial^2 i}{\partial x^2} = \frac{\mu}{\rho}\frac{\partial i}{\partial t}$$

is a form of equation characteristic of diffusion or heat flow; the corresponding wave equation is

$$\frac{\partial^2 i}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 i}{\partial t^2}$$

as in Chap. 14.

$1/e$ times the current density at the surface, or roughly $\frac{1}{3}$; the vector at $x = 2\delta$ is about $\frac{1}{3}$ of that at $x = \delta$, and so on. Similarly, the phase angle of current density is 1 radian (57.3 degrees) more lagging at each successive increment of depth δ. Figure 7-7 shows these relations, and is generally applicable to any frequency and any kind of conducting material.

It will be seen that at a depth of about 3δ (at $x = \pi\delta$, to be exact) the current density is totally out of phase with the surface current density. Current at this depth is actually flowing in the opposite direction from current at the surface. This, of course, is wasteful for all practical purposes. However, current density at this depth is fortunately small. It will also be seen that the total amount of current flowing at depths greater than 4δ or 5δ is so small as to be negligible for most purposes.

6. Effective Alternating-current Resistance. There are two ways of thinking of skin effect. One is to think of it as merely a nonuniform distribution of current in the conductor. The other way is to think of it as a circulating current within the conductor superimposed on a uniform current density. From either point of view, it will be seen that skin effect increases the resistive loss of energy in the conductor. This suggests that a higher value of *effective* resistance is needed to account for the greater loss. We find the value of effective resistance as follows.

The impedance of a length of conductor is $Z = V/I$, and we define the effective resistance by writing

$$Z = R_{ac} + jX = \frac{V}{I} \tag{7-36}$$

where V is the rms voltage drop in the given length of conductor
I is the rms current in the conductor, the integral of current density over the conductor cross section

In general, V, I, and Z are complex, and R_{ac} is defined as the real component of Z:

$$R_{ac} = \text{real part of } \frac{V}{I} \tag{7-37}$$

(It follows from this definition that power lost in resistance in the conductor is $|I|^2R_{ac}$.)

To compute R_{ac} of a conductor, it is first necessary to know the current, which is found by integrating the current density as in Eq. 7-2. In a semi-infinite block, the total current is infinite, but we can find the current in a section of the block, such as the section of width u indicated in Fig. 7-5. Current through the rectangle $MNN'M'$ is $iu\,dx$, and integrating from the surface to an infinite depth gives

$$I = \int_0^\infty iu\,dx \tag{7-38}$$

Using Eq. 7-34 for $\dot{i}$, and writing γ for $\sqrt{\omega\mu/2\rho}\,(1 + j)$, as in Eq. 7-32,

$$I = \int_0^\infty \dot{i}_0 e^{-\gamma x} u \, dx = \frac{u\dot{i}_0}{\gamma} \tag{7-39}$$

The next step is to determine the voltage drop along the conductor in the direction of current flow. Since we know that the drop must be the same along all filaments within the conductor between any two cross-sectional surfaces, it is only necessary to find the voltage drop along any one filament, and the voltage drop along every other filament must be the same.

The total voltage drop along any filament may be conveniently considered in three parts: (1) the resistive drop, (2) the reactive drop owing to magnetic field that exists within the conductor material, and (3) the reactive drop owing to magnetic field that exists in space around the conductor. If the conductor is a round wire, part 2 takes into account magnetic flux lines within the conductor that surround the filament in question—the flux lines, that is, that lie between the filament and the surface of the conductor. Part 3 takes into account the flux lines that circle about the conductor and link with *all filaments* of the conductor. Since the magnetic field of part 3 affects all filaments of the conductor equally, inducing the same reactive voltage in each, this part has nothing to do with skin effect. Let us therefore define a voltage drop per unit length of conductor, to be called V_i, as the sum of (1) the resistive drop and (2) the *internal* reactive drop (owing to magnetic field within the conductor material). Like the total voltage drop, this quantity V_i must be the same along every filament of the conductor.

V_i is most easily determined for a filament just at the surface of the conductor, for such a filament is not surrounded by any of the *internal* magnetic field, and V_i is completely given by the resistive drop: at the surface,

$$V_i = \rho \dot{i}_0 \tag{7-40}$$

This reasoning is readily extended to a semi-infinite conductor by considering it to be a round wire that has been expanded without limit. Equation 7-40 applies to a semi-infinite conductor as well as to a round wire.

Impedance is now found, by Eq. 7-36, as the ratio of voltage to current. Let us consider the impedance of a block of the semi-infinite conductor of Fig. 7-5 that is one unit wide ($u = 1$). By Eq. 7-39 the current carried in such a block of unit width is

$$I = \frac{\dot{i}_0}{\gamma} \tag{7-41}$$

The voltage per unit length along any filament of such a block, including resistance and internal inductance, is given by Eq. 7-40. The internal impedance per unit length of such a block is the ratio

$$z_i = \frac{V_i}{I} = \frac{\rho i_0}{i_0/\gamma} = \rho\gamma = \sqrt{\frac{\omega\mu\rho}{2}}\,(1 + j) \qquad (7\text{-}42)$$

The effective a-c resistance per unit length of such a block is, by Eq. 7-37, the real component of z_i:

$$r_{ac} = \sqrt{\frac{\omega\mu\rho}{2}} \qquad (7\text{-}43)$$

We have now succeeded in expressing the a-c resistance per meter length of a meter-wide section of conducting material that is part of a semi-infinite block. Resistance is given in terms of the resistivity and permeability of the conducting material and the frequency of the current. When the phrase "surface resistivity" is used in connection with skin effect, it is this quantity r_{ac} that is meant.

The Rayleigh Approximation. Consider a layer of material, 1 meter wide, at the surface of the conductor, the thickness of the layer being δ. The meaning of δ is the same as in Eq. 7-35. Consider this layer to be carrying *direct* current. Because it is 1 meter wide, the d-c resistance per unit length of such a layer is ρ/δ. Introducing the value of δ from Eq. 7-35, the resistance of a layer of thickness δ is $\sqrt{\omega\mu\rho/2}$. By comparison with Eq. 7-43, we can make the following statement.

The effective a-c resistance of a very large conductor is the same as the d-c resistance of a superficial layer of this conductor of thickness δ where $\delta = \sqrt{2\rho/\omega\mu}$. We have proved this statement to be exact for a semi-infinite conductor. It is an excellent approximation (known as the Rayleigh approximation) if a conductor is so large, and the frequency so high, that all dimensions of the conductor are many times larger than δ. Another way of stating this requirement is that the conductor must have no sharp corners, no very thin sections, and the radius of curvature of its surface must be everywhere many times δ. It is also necessary, for the Rayleigh approximation to be good, to have approximately uniform current density over all the surface of the conductor; this will automatically result from symmetry in large round wires, but with a conductor the shape of a railroad rail, for instance, the approximation is less satisfactory.

For convenience, δ is often called the "equivalent depth of penetration," or merely the "skin depth" of a conductor. This must *not* be understood to mean that current is really limited to a surface layer of depth δ, for there is actually appreciable current density at several times this depth, as shown in Fig. 7-7.

7. Current Distribution in a Round Wire. In Sec. 3 we derived an equation for current density within a round wire. This led to Eq. 7-18, a differential equation, for which a solution is required. The form of this equation is somewhat less familiar than that of Eq. 7-29, which was shown to have an exponential solution, but it is well known to mathematicians and its solution is called a Bessel function.[4] The solution of

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[4] The Bessel function of the first kind and zero order is indicated by the symbol J_0, and is defined by the infinite series

$$J_0(nx) = 1 - \frac{(nx)^2}{2^2} + \frac{(nx)^4}{2^4(2!)^2} - \frac{(nx)^6}{2^6(3!)^2} + \cdots \tag{7-44}$$

This function is important in many physical problems besides skin effect, the classic example being vibration of a round diaphragm such as the head of a drum. There are several related functions; let us evaluate

$$\frac{n}{x}\int_0^x xJ_0(nx)\,dx \tag{7-45}$$

This integration is readily carried out, term by term, on Eq. 7-44, and using the symbol $J_1(nx)$ to indicate the new function that results, we have

$$J_1(nx) = \frac{nx}{2}\left(1 - \frac{(nx)^2}{2^2 1!2!} + \frac{(nx)^4}{2^4 2!3!} - \frac{(nx)^6}{2^6 3!4!} + \cdots\right) \tag{7-46}$$

This new function is called a Bessel function of the first kind and first order, and from the above equations it is related to J_0 by

$$J_1(nx) = \frac{n}{x}\int_0^x xJ_0(nx)\,dx \tag{7-47}$$

The reader should not be disturbed by the appearance of new functions. From time to time, a physical problem finds mathematical expression in a differential equation that has no solution in terms of known functions; the solution can be expressed as an infinite series, but not otherwise. Numerical results can be computed from the infinite series, but this is tedious; therefore, if the problem is of enough importance to justify the work, tables of values are computed. Thus a new function is born, and it is given a name to identify it and a symbol for convenient use. Friedrich Wilhelm Bessel was a German astronomer (1784-1846); in 1817 he introduced the functions now known as Bessel functions in the solution of a problem relating to motion of the planets. In the next few years he studied the properties of these functions, and computed tables of values. It is notable that earlier scientists had solved the differential equation that leads to Bessel functions (Bernoulli in 1732 for the problem of a swinging chain and Euler in 1764 for a stretched membrane) but Bessel was the first to study the functions systematically.

There is nothing to prevent the reader from developing a new function. You may take a differential equation, find the solution as a series, compute numerical values, and tabulate them. This new function you can then name after yourself. If it is sufficiently useful to enough people it will, perhaps, survive.

Eq. 7-18 is written

$$i = i_0 J_0(nr) \tag{7-48}$$

where

$$n = \sqrt{\frac{-j\omega\mu}{\rho}} = \sqrt{\frac{\omega\mu}{2\rho}}(-1 + j)$$

and i_0 is current density at the center of the round wire. $J_0(nr)$ is the Bessel function defined by Eq. 7-44. Expanding, it may be written

$$i = i_0 J_0\left[\sqrt{\frac{\omega\mu}{2\rho}}(-1 + j)r\right] \tag{7-49}$$

To show that this is a correct solution, Eq. 7-48 can be substituted for i in Eq. 7-18, expanding the Bessel function in series form, and an identity results.

Equation 7-49 for current distribution in a round wire is comparable to Eq. 7-34 for current distribution in a semi-infinite conductor. For the round wire, however, a Bessel function is used instead of an exponential function.[5] We may of course expect that if frequency is very high, or the conductor very large, so that skin effect is extreme, the Bessel-function solution will become indistinguishable from the exponential solution. Indeed, Figs. 7-6 and 7-7 are reasonably good representations of current distribution in a round wire if the radius of the wire is several times the equivalent depth of penetration δ. Current density near the center of a round wire is greater than is indicated in these figures, however; it may be considered that current diffuses toward the center of a round wire from all sides, instead of from one direction only, thereby strengthening the flow of current near the center. Figure 7-8 is drawn to show current distribution in a round wire of such a size that $r = 3.6\delta$. It is

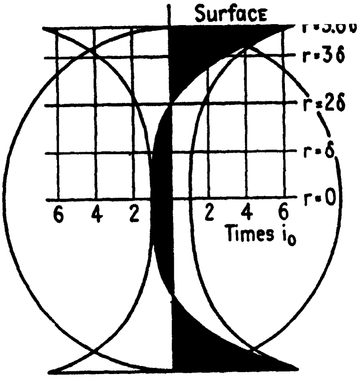

FIG. 7-8. Current density in a round wire.

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[5] The complete solution of Eq. 7-18 contains a Bessel function of the second kind, as well as the function of the first kind given in Eq. 7-48. This is analogous to the first of the exponential terms of Eq. 7-33, and vanishes for the same reason; *i.e.*, the Bessel function of the second kind is infinite for zero argument, and gives infinite current density at the center of the round wire unless its coefficient is zero. Infinite current density is impossible, for it would require infinite voltage. Hence Eq. 7-48 is correct for a solid round wire. For a tubular conductor, the Bessel function of the second kind does not vanish.

comparable to the first of the curves of Fig. 7-6. Although the distribution of current density is shown at only one instant, envelopes are indicated within which all current-density curves will lie; for other instants of time, the current-density wave of Fig. 7-8 can be imagined to move inward from the surface while remaining within the indicated envelope.

The size of conductor shown in Fig. 7-8 is such that the surface current density is just out of phase with the center current density. This is convenient for illustration, but is not important. For a smaller wire of any size, merely draw a correspondingly smaller circle in the diagram, and disregard the parts of the curve above and below that circle.

8. Skin-effect Resistance Ratio. The effective resistance of a section of a semi-infinite conductor was computed in a previous section. We are now prepared to find the effective resistance of a round wire. The same steps will be required in the solution.

Total current in a round wire of radius a is, as in Eq. 7-10,

$$I = \int_0^a i \cdot 2\pi r \, dr \tag{7-50}$$

Introducing Eq. 7-48 for current,

$$I = 2\pi i_0 \int_0^a rJ_0(nr)dr \tag{7-51}$$

We now make use of Eq. 7-47 and write

$$I = 2\pi i_0 \frac{a}{n} J_1(na) \tag{7-52}$$

for the total current in the wire.

Voltage drop along a filament of unit length at the surface of the wire, considering resistance and internal reactance, but not external reactance, is

$$V_i = \rho i_{(\text{surface})} = \rho i_0 J_0(na) \tag{7-53}$$

where a is the outer radius of the conductor.

The internal impedance per unit length of the round wire is therefore

$$z_i = \frac{V_i}{I} = \frac{n\rho J_0(na)}{2\pi a J_1(na)} \tag{7-54}$$

The effective a-c resistance per unit length of the wire is the real component of z_i.

A useful practical value is the ratio of effective a-c resistance to the d-c resistance of the wire. The d-c resistance per unit length of a round conductor of radius a is

$$r_{dc} = \frac{\rho}{\pi a^2} \tag{7-55}$$

and therefore the so-called *skin-effect resistance ratio* is

$$\frac{r_{ac}}{r_{dc}} = \text{real part of } \frac{naJ_0(na)}{2J_1(na)} \tag{7-56}$$

This ratio obviously applies to any length of wire as well as to unit length, and it is commonly designated R/R_{dc} or R/R_0.

Computation of numerical values from Eq. 7-56 has been done by various people. Most published values are based on tables prepared by the National Bureau of Standards (*Scientific Paper* 169, *National Bureau of Standards*, 1912, Table XXII), and the accompanying Table 7-3 is from that source. The first column of this table gives the quantity *ma*;

TABLE 7-3

SKIN-EFFECT RESISTANCE RATIO AND INTERNAL INDUCTANCE RATIO FOR SOLID ROUND WIRES

$ma = \sqrt{2}\,\frac{a}{\delta}$	$\frac{R_{ac}}{R_{dc}}$	$\frac{L_{ac}}{L_{dc}}$*	$ma = \sqrt{2}\,\frac{a}{\delta}$	$\frac{R_{ac}}{R_{dc}}$	$\frac{L_{ac}}{L_{dc}}$
0.0	1.000	1.000	3.0	1.318	0.845
0.2	1.000	1.000	3.2	1.385	0.814
0.4	1.000	1.000	3.4	1.456	0.782
0.6	1.001	1.000	3.6	1.529	0.749
0.8	1.002	0.999	3.8	1.603	0.717
1.0	1.005	0.997	4.0	1.678	0.686
1.2	1.011	0.995	5.0	2.043	0.556
1.4	1.020	0.990	6.0	2.394	0.465
1.6	1.033	0.983	7.0	2.743	0.400
1.8	1.052	0.974	8.0	3.094	0.351
2.0	1.078	0.961	10.0	3.799	0.282
2.2	1.111	0.945	20.0	7.328	0.141
2.4	1.152	0.925	30.0	10.861	0.094
2.6	1.201	0.901	40.0	14.395	0.071
2.8	1.256	0.875	50.0	17.930	0.057

*Ratio of *internal* inductances; equal to $K\varphi$ for nonmagnetic wires.

a is the radius of the wire and *m* is the magnitude of *n* of Eq. 7-56. By definition, $m = \sqrt{\omega\mu/\rho}$, and it may be seen from Eq. 7-48 that therefore $n = \sqrt{-j}\, m$. Also, from Eq. 7-35, we may write $ma = \sqrt{2}\, a/\delta$, where δ is the equivalent depth of penetration; this is the most convenient way to find *ma* if the depth of penetration is known.

Practical Determination of Skin-effect Resistance Ratio. The method to use to find the a-c resistance of a given conductor depends on the accuracy required, and on the frequency. Frequency and size are related by the ratio of conductor radius to skin depth, a/δ. If a/δ is less than 1, the error that results from neglecting skin effect is less than 2 per cent.

Table. For highest accuracy, use Table 7-3. However, at rather high radio frequency, a/δ may exceed 50 and hence be beyond the accurate range of the table.

Very-high-frequency approximation. If a/δ is greater than 50 (or for a rough approximation with a/δ greater than 5), use

$$\frac{R_{ac}}{R_{dc}} = \frac{1}{2}\frac{a}{\delta} \tag{7-57}$$

where a is radius of the round wire (meters)

$\delta = \sqrt{2\rho/\omega\mu} = \sqrt{\rho/\pi f\mu}$ (units as in Eq. 7-35)

This formula is based on the Rayleigh approximation, as follows. R_{dc} per unit length $= \rho/\pi a^2$. R_{ac} per unit length is ρ divided by the cross-sectional area of a surface layer of thickness δ. Approximately, $R_{ac} = \rho/2\pi a\delta$. Dividing, Eq. 7-57 results.

This approximation has an error of less than 1 per cent if a/δ is greater than 50, and less than 10 per cent if a/δ is greater than 5. See also Eqs. 12-16 and 12-17, which have the same accuracy.

High-frequency approximation. If good accuracy is required, and a/δ is between 4 and 50, use

$$\frac{R_{ac}}{R_{dc}} = \frac{1}{2\dfrac{\delta}{a} - \left(\dfrac{\delta}{a}\right)^2} \tag{7-58}$$

For meaning of a and δ, see Eq. 7-57.

The error is less than 1 per cent if a/δ is greater than 4.

This formula is also based on the Rayleigh approximation, but uses an exact expression for the cross-sectional area of the surface layer of depth δ. For unit length of the round wire, $R_{ac} = \rho/[\pi a^2 - \pi(a - \delta)^2]$; as before, $R_{dc} = \rho/\pi a^2$; and division gives Eq. 7-58.

Charts. The charts of Fig. 7-9 are useful (1) *for a quick and easy approximation of skin effect at any frequency* up to $a/\delta = 50$ (above that, use Eq. 12-16 or 12-17) and (2) *for accurate results if a/δ lies between* 1 *and* 4. The upper chart of Fig. 7-9 gives the reciprocal of the skin-effect resistance ratio, R_{dc}/R_{ac}, as a function of a/δ, and the lower chart permits quick determination of a/δ from the frequency and the size of wire.

As an example, let us find the skin-effect resistance ratio of No. 20

copper wire at 1 megacycle frequency. Entering the lower chart at a frequency of 10^6 cycles per second, read a/δ on the sloping line marked "No. 20." Follow this value of a/δ, which is about 6, to the upper chart, where R_{dc}/R_{ac} is found to be about 0.3. The a-c resistance is therefore about 3.3 times the d-c resistance.

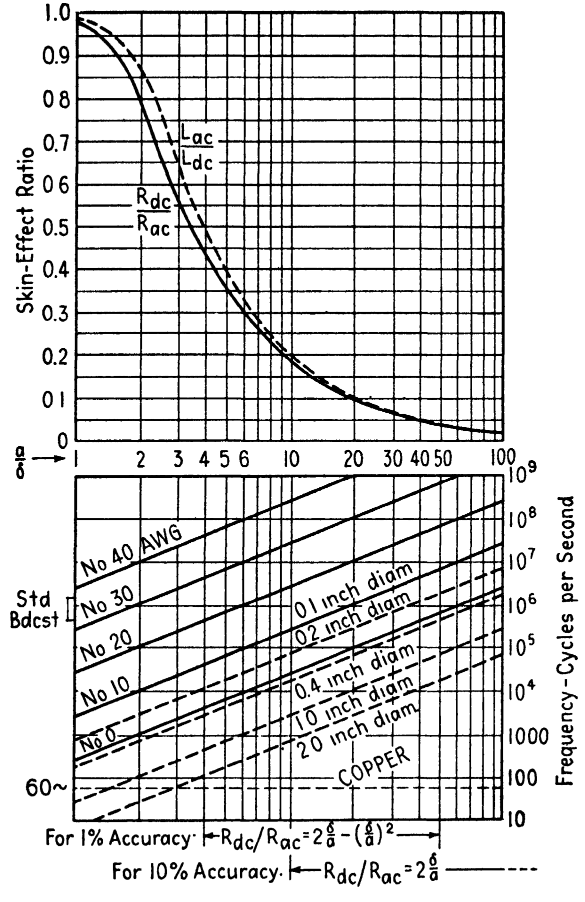

FIG. 7-9. Skin-effect resistance ratio and internal inductance ratio for round wires. Enter the lower chart at right with frequency, move horizontally to appropriate wire size, move vertically to curve on upper chart, and read skin-effect ratio at left.

The upper chart is for any metal, but the lower chart is for copper only. To use the lower chart for other metals, multiply the actual frequency by the following factor (relative conductivity) before entering the chart: Aluminum, 0.61; Brass, 0.25; Gold, 0.71; Silver, 1.06; Tin, 0.15.

a = radius of wire, meters

δ = "skin depth" = $0.564\sqrt{\rho/\mu f}$ meters

Ranges of a/δ in which approximate formulas are suitable are indicated at the bottom of the chart.

Although the lower chart of Fig. 7-9 is specifically for copper conductors, it can be used for conductors of any nonmagnetic material by the following artifice. The skin depth δ is a function of the ratio of resistivity to frequency; if this ratio is correct, the chart will give the correct value for δ. Hence to use the chart for aluminum wire, the conductivity of which is 0.61 times that of copper (see Table 7-2), multiply the actual frequency by 0.61, and enter the chart with this modified value for frequency. The physical reason is that copper, because its resistivity is lower than that of aluminum, has the same current density distribution, and hence the same skin effect, at a proportionately lower frequency.

The accuracy obtainable from charts is not great but, considering the uncertainty that cannot be avoided in temperature and resistivity data, the charts are often good enough.

9. Skin-effect Inductance Ratio. The redistribution of current resulting from skin effect alters the inductance of a conductor as well as its resistance. It alters the internal inductance only, for skin effect changes the magnetic field pattern *within* the conductor. Since the magnetic field outside the conductor is unchanged by skin effect, the external inductance is unaffected.

Table I includes formulas for computing transmission-line inductance. In each inductance formula, there is a term that gives the internal inductance (see Chap. 6). In each formula, this term is multiplied by K_φ, called a permeability and skin-effect constant. We can now evaluate K_φ.

Equation 7-54 gives the internal impedance per unit length of a round wire, and Eq. 7-42 gives the internal impedance of a unit section of a semi-infinite conductor. For each type of conductor, we found the effective a-c resistance as the real component of this expression for impedance. Correspondingly, the true internal reactance of the conductor is the imaginary component of this expression for impedance. Internal inductance is this internal reactance divided by ω.

The imaginary component of Eq. 7-42 is obvious. Indeed, the internal reactance of the semi-infinite conductor is numerically equal to its effective resistance as given in Eq. 7-43. When a practical conductor is considered to be semi-infinite, using the Rayleigh approximation, this internal reactance is so small compared with the external reactance that it is entirely neglected. It need not be further discussed here.

The imaginary component of the internal impedance of a round wire gives the internal reactance of the wire. The internal inductance of the wire is the internal reactance divided by ω. From Eq. 7-54,

$$l_i = \text{imaginary part of } \frac{n\rho J_0(na)}{2\pi\omega a J_1(na)} \tag{7-59}$$

The d-c internal inductance of a round wire is, by Eq. 6-14,[6] equal to $\mu/8\pi$. Dividing, the skin-effect inductance ratio is obtained:

$$\frac{l_{i(ac)}}{l_{i(dc)}} = \text{imaginary part of } \frac{4n\rho J_0(na)}{\omega\mu a J_1(na)} \tag{7-60}$$

Numerical values of the skin-effect inductance ratio have been computed from Eq. 7-60, and are included in Table 7-3. Values good enough for almost all practical purposes may be taken from a curve of Fig. 7-9. The skin-effect inductance ratio need not be found very accurately, for it applies to the internal inductance only, and the internal inductance is hardly ever more than a small part of the total inductance.

As mentioned above, the formulas of Table 1 contain a factor K_φ in each term for internal inductance. It was shown by Eq. 6-17*a* that K_φ is equal to the relative permeability μ_r if a conductor is carrying direct current. We now see that skin effect should be included in the meaning of K_φ, for

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[6] Equation 6-14 is derived using the unsatisfactory concept of *partial flux linkages*. The *partial flux linkage* concept implies that a certain amount of flux linking half the current in a wire is equivalent to half as much flux linking all the current—a conclusion obviously untrue as regards current distribution, and open to suspicion as regards induced voltage. This oversimplified treatment can now be replaced by the following derivation, which takes the true current distribution into account.

From Eq. 7-54,

$$z_i = \frac{n\rho J_0(na)}{2\pi a J_1(na)} \tag{7-61}$$

Expanding the Bessel functions in series, from Eqs. 7-44 and 7-48, and using

$$n = \sqrt{-j\frac{\omega\mu}{\rho}} \tag{7-62}$$

we obtain

$$z_i = \frac{\rho}{\pi a^2} \frac{1 + j\dfrac{\omega\mu a^2}{4\rho} + \cdots}{1 + j\dfrac{\omega\mu a^2}{8\rho} + \cdots} \tag{7-63}$$

Then

$$l_i = \text{imaginary part of } \frac{z_i}{\omega} = \frac{\rho}{\pi a^2 \omega} \frac{1 + j\dfrac{\omega\mu a^2}{4\rho} + \cdots}{1 + j\dfrac{\omega\mu a^2}{8\rho} + \cdots} \tag{7-64}$$

To find the d-c inductance, let ω approach zero. For very small values of ω, all but the first two terms of each series of Eq. 7-64 can be neglected, for they contain increasing powers of ω. The remaining fraction is rationalized by multiplying numerator and denominator by $1 - j\omega\mu a^2/8\rho$. The denominator will then contain a with alternating current the internal inductance is reduced from its d-c value by the skin-effect inductance ratio. Hence, with alternating current,

$$K_\varphi = \mu_r \frac{l_{i(ac)}}{l_{i(dc)}} \tag{7-68}$$

It is from this formula that K_φ is to be evaluated. For copper or aluminum wires, or any ordinary wires except iron and steel, $\mu_r = 1$, and K_φ is identical with the skin-effect inductance ratio. For iron and steel wires, the permeability is so variable and uncertain that computation is quite unsatisfactory.[7]

For practical determination of K_φ, for use in inductance formulas, the following rules will give a degree of accuracy that is probably sufficient (the error will not exceed 10 per cent of $l_{i(dc)}$, and will be a much smaller part of the total inductance). Find the ratio of the radius of the wire to the equivalent skin depth, a/δ (Fig. 7-9 gives this ratio for certain sizes).

1. If a/δ is less than 1.8, use $K_\varphi = 1$.
2. If a/δ is between 1.8 and 20, read K_φ from Fig. 7-9.
3. If a/δ is greater than 20, use $K_\varphi = 0$.

10. Insulation Loss. Besides the loss that results in a transmission line from current in the conductor, there is also loss resulting from the application of electric field to the insulation. Loss of power in the insulation can occur in a number of ways, of which the following are the most important.

Actual conductivity of the material separating one conductor of the

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term 1, and another real term in ω^2 which is negligible compared to 1; the denominator therefore approaches 1 as ω approaches zero. The numerator will contain a real term and an imaginary term; we are interested only in the imaginary term, which is $\omega\mu a^2/8\rho$. This combines with the coefficient of Eq. 7-64 to give

$$\frac{\rho}{\pi a^2 \omega} \frac{\omega \mu a^2}{8\rho} \tag{7-66}$$

the limit of which, as ω approaches zero, is the d-c internal inductance:

$$l_{i(dc)} = \frac{\mu}{8\pi} \tag{7-67}$$

Since this agrees with Eq. 6-14, Eq. 6-14 is justified.

(The limit of the real part of z_i can similarly be found to be the d-c resistance of the conductor, and this is an interesting check on the algebra but gives no new information.)

[7] Certain test results are given in the Westinghouse "Electrical Transmission and Distribution Reference Book" (Ref. 18). For steel wires coated with copper (Copperclad, Copperweld, and other trade names), it is usually satisfactory to assume that the current is confined to the copper. Both the high resistivity of the steel, and the extreme skin effect resulting from its high permeability, tend to make this true.

transmission line from another is possible. If a cable were made with sufficiently poor insulation, conduction from wire to wire would take place. This type of conduction would ideally be accounted for by a value of g in the transmission-line equations that would be independent of either frequency or voltage. Loss from pure conductivity of insulation undoubtedly occurs but, if the amount of loss is significant, other causes almost always predominate.

When telephone lines and power lines are suspended in the air (*open-wire lines*) by means of porcelain or glass insulators on poles, or on steel towers, the insulators are in danger of becoming wet and dirty. There will then be electrical leakage through the conducting film on the insulator surface. Pure rain water and clean snow are practically nonconducting, but the presence of any salt, or even the soluble material found in ordinary dust, is enough to cause significant loss. Such *surface leakage* loss comes close to being ideal conductivity as described in the preceding paragraph, except that it takes place at the points of support only, and is not uniformly distributed along the line. This makes little difference in analysis; the much more important and obvious difficulty is that this loss is a temporary and changeable kind of thing. Ordinarily the best that can be done is to say that the appropriate value of g on a telephone line, for instance, will probably lie in a range that corresponds to a certain type of weather.

High-voltage power lines are subject to loss of another nature. If the electric gradient in the air close around the high-voltage conductors exceeds a critical value (about 30,000 volts per centimeter), there is ionization of the air. The resulting positive and negative ions are propelled by the electric field and constitute an electric current through the air. This phenomenon, because of the luminosity that accompanies it, is called corona. *Corona loss* is dependent in a complicated manner on both voltage and frequency. Corona will be mentioned again in the discussion of power lines, Chap. 11. It can be approximately accounted for at any particular voltage and at one particular frequency by assigning an equivalent value of g to represent the amount of the corona loss, but such an application is so restricted that it is rarely of much value. Fortunately for purposes of analysis, power lines are ordinarily operated at voltages that give little, if any, corona loss.

When an alternating electric field is applied to insulating material, there is a loss of energy sometimes known as *dielectric hysteresis*. As the electric field in the dielectric material increases, energy is stored in the capacitance of the system,[8] and as the electric field diminishes in the next quarter cycle the stored energy is returned to the circuit—but a small part of the stored energy is not recovered, being lost in the form of heat.

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[8] A brief discussion of polarization is given in Chap. IV of "Fundamentals of Electric Waves" (Ref. 29).

The implication of the name is that there is some kind of frictional action; if we may assume that energy is stored in the dielectric material through some stretching, distortion, or displacement of the atoms or molecules, it may be that imperfect elasticity is responsible for the loss. Such an explanation is oversimplified, no doubt; however, it is consistent with the chief characteristic of dielectric hysteresis, which is that the power loss is very nearly proportional to the frequency. That is, a certain amount of energy is lost in each cycle.

This proportionality to frequency distinguishes dielectric hysteresis from conduction loss. Dielectric hysteresis is negligibly small at low frequency, but at high radio frequencies it is the predominant type of insulation loss. It is characteristic of all solid and liquid insulating materials, but since it is extraordinarily low in mica, quartz, and various polymerized synthetic compounds, these substances are useful for microwave insulation. Polyethylene, polystyrene, and similar materials are specially valuable. These will be discussed again in Chap. 12 on high-frequency transmission lines.

Dielectric hysteresis is readily included in the transmission-line equations. This is done by introducing an equivalent value of g to represent the power loss. The proper value of g to use is dependent on frequency, but if the transmission-line equations are used for only one frequency at a time, or for a narrow band of frequencies, this is not a serious limitation.

To find the appropriate value of g to use to represent dielectric loss, it is first necessary to have information about the characteristics of the insulating material. There are various ways of expressing data about dielectric loss for tabulation in handbooks and similar reference tables. One of the most widely used figures is the so-called *power factor of the material*. Reference books that give information in other forms[9] commonly tell how their data are related to the *power factor*. The power factor of an insulating material is defined as the power factor of an otherwise perfect condenser using that material as its dielectric.

In a transmission line, the power factor of the dielectric material relates g, the conductance per unit length, to c, the capacitance per unit length. For example, consider a length of transmission line 1 meter long, a cross section of which is shown in Fig. 7-10. (A coaxial line is shown, but this is for illustration only, and any type of line would be equally satisfactory.) This unit length of line is isolated and is acting as a condenser. The capacitance between the two line conductors is c, and the dielectric loss is to be represented by an equivalent conductance g. Admittance of this section acting as a condenser is

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[9] The *loss factor*, also widely used, is $1/\rho\omega\epsilon_0$ or, with the same degree of approximation used in Eq. 7-71, it is the power factor of the material times the relative dielectric constant. The power factor may also be referred to as the *loss tangent* if it is desired to emphasize that it is truly the tangent and not the sine of the *loss angle*. If loss in a dielectric material is accounted for by using a complex dielectric constant $\epsilon = \epsilon' - j\epsilon''$, then ϵ' is the usual dielectric constant and ϵ''/ϵ' is the loss tangent.

$$y = g + j\omega c \tag{7-69}$$

The power factor of such a condenser is

$$\text{Power factor} = \frac{g}{\sqrt{(\omega c)^2 + g^2}} \tag{7-70}$$

Since we are working with good dielectric materials, and hence with very small power factors, g^2 is very much smaller than $(\omega c)^2$. It is an excellent approximation to neglect g in the denominator of Eq. 7-70 and write

$$\text{Power factor} = \frac{g}{\omega c} \tag{7-71}$$

This quantity is the power factor of the line acting as a condenser; it is therefore (by definition) the power factor of the insulating material, and thus we have

$$g = \omega c \times (\text{power factor}) \tag{7-72}$$

Hence, to find the value of g to use for a transmission line with dielectric loss, read the power factor of its insulation from a table, and use Eq. 7-72.

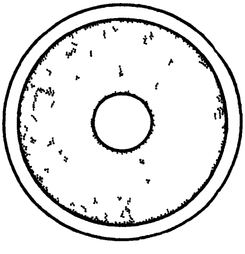

FIG. 7-10 A line with solid dielectric insulation.

The power factor of a material is a convenient means of expressing dielectric loss because the power factor is reasonably constant over a fairly wide range of frequency. In the ultra-high-frequency range (wherein dielectric loss is most important), dielectric power factors range from the order of 0.0005 for mica, quartz, and polystyrene, to 0.005 for glass and steatite, and to 0.05 for phenol products.

For ultra-high-frequency transmission lines with solid dielectric, attenuation of the transmitted signal is determined largely by dielectric loss. It is therefore convenient to compute the attenuation α directly in terms of the dielectric power factor. Even better, loss can be given in decibels per unit length. Formulas for this purpose are derived in Chap. 12.

11. Experimental Determination of Constants. Often, as with a twisted-pair transmission line, or the circuit of two wires within a telephone cable, it is impossible to compute the constants r, l, g, and c from dimensions. On many other lines it is not convenient to compute them, though it might be possible. Moreover, if a line has been designed by computation, it is desirable to measure the constants when the line is completed, to ensure that line construction is satisfactory.

Theoretically, measurement of any four different currents or voltages on the transmission line gives the necessary information to permit solution for the four constants. Practically, the following impedance measurements are usually the most convenient.

At some known frequency, measure input impedance to the line with the receiving end of the line open-circuited. At the same frequency, measure input impedance to the line with the receiving end short-circuited. The measurements should be made, as with an impedance bridge, to give resistance and reactance components of input impedance with both terminations, thereby providing four measured quantities.

The sending-end impedance with the receiving end open may be designated $Z_{s(op)}$, and the sending-end impedance with the receiving end short-circuited, $Z_{s(sh)}$. Using Eqs. 5-30 and 5-32,

$$Z_{s(op)} = -jZ_0 \cot Bx \tag{7-73}$$

$$Z_{s(sh)} = jZ_0 \tan Bx \tag{7-74}$$

The length of the line, x, is known. B and Z_0 are to be found.

Multiplying the two measurements together and taking the square root,

$$Z_0 = \sqrt{Z_{s(op)} \cdot Z_{s(sh)}} \tag{7-75}$$

Dividing Eq. 7-74 by 7-73,

$$(\tan Bx)^2 = \frac{Z_{s(sh)}}{-Z_{s(op)}} \tag{7-76}$$

from which

$$Bx = \tan^{-1} \sqrt{\frac{Z_{s(sh)}}{-Z_{s(op)}}} \tag{7-77}$$

B can then be found, x being known.[10] An alternative form is

$$\gamma x = \tanh^{-1} \sqrt{\frac{Z_{s(sh)}}{Z_{s(op)}}} \tag{7-78}$$

There is an uncertainty about the magnitude of the real component of Bx. It must be remembered that adding π to a quantity does not change its tangent (Formula 45 of Table II). The inverse tangent is a multivalued function, and any multiple of π may be added or subtracted. (This is the meaning of k in Formula 50 of Table II.) Physically, k is the number of complete half-wavelengths in the length of the line. If the line is less than 1 half-wavelength, $k = 0$. If the line is between 3 and 4 half-wavelengths, for example, $k = 3$. In general, k is the integer part of

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[10] Equation 7-77 requires the evaluation of the inverse tangent of a complex quantity. The inverse tangent of a real number is easily found from a table of tangents. If a table or chart of the tangent of a complex argument is available, it can be used in a similar way to find the inverse tangent. However, Formula 50 of

$$\frac{2fx}{\text{Velocity}} \tag{7-81}$$

where f is frequency and x is length of line.

In every practical problem, the correct value of k must be determined from physical information. Equation 7-81 is useful if the velocity of propagation is known. On an open-wire line the velocity is somewhere near 180,000 miles per second, or a few per cent less than 3×10^8 meters per second. If the line is a cable, an approximate value of velocity may be available from computation or previous experience.

If it is impossible thus to estimate the velocity and so determine the correct value of k, test measurements of $Z_{s(op)}$ and $Z_{s(sh)}$ may be repeated at another frequency, and there will in general be only one solution consistent with both sets of test data. In particularly difficult cases, impedances can be measured over a range of frequency. By plotting the

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Table II is likely to be more convenient; it looks formidable, but is easily applied.

An alternative way to find Bx is to use either the inverse sine or the inverse cosine, instead of the inverse tangent of Eq. 7-77. Formula 52 of Table II gives

$$Bx = \tan^{-1}\sqrt{\frac{Z_{s(sh)}}{-Z_{s(op)}}} = \sin^{-1}\sqrt{\frac{Z_{s(sh)}}{Z_{s(sh)} - Z_{s(op)}}}$$

$$= \cos^{-1}\sqrt{\frac{Z_{s(op)}}{Z_{s(op)} - Z_{s(sh)}}} \tag{7-79}$$

The inverse sine can be evaluated by using Formula 49, or the inverse cosine by using Formulas 51 and 49, of Table II. These formulas offer no advantages over the use of Eq. 7-77 and Formula 50, but if the approximate formulas of Sec. 6, Chap. 8, are applicable, they give a particularly easy means of evaluation. The inverse sine and cosine can also be evaluated from the chart of Fig. 4-5, or any equivalent chart, if the accuracy of a reading from a chart is adequate.

Hyperbolic relations similar to Eq. 7-79 are

$$\gamma x = \tanh^{-1}\sqrt{\frac{Z_{s(sh)}}{Z_{s(op)}}} = \sinh^{-1}\sqrt{\frac{Z_{s(sh)}}{Z_{s(op)} - Z_{s(sh)}}}$$

$$= \cosh^{-1}\sqrt{\frac{Z_{s(op)}}{Z_{s(op)} - Z_{s(sh)}}} \tag{7-80}$$

results, and noting the frequencies at which maximum and minimum impedances occur, the number of half-wavelengths in the line can readily be deduced.

Having found Z_0 and Bx, z and y are next computed. From Bx (or γx), find jB or γ. Then, since

$$Z_0 = \sqrt{\frac{z}{y}} \quad \text{and} \quad jB = \gamma = \sqrt{zy} \tag{7-82}$$

we find z by multiplying

$$z = \sqrt{\frac{z}{y}} \cdot \sqrt{zy} = Z_0(jB) = Z_0\gamma \tag{7-83}$$

and y by dividing

$$y = \frac{\sqrt{zy}}{\sqrt{\frac{z}{y}}} = \frac{jB}{Z_0} = \frac{\gamma}{Z_0} \tag{7-84}$$

Finally, r, l, g, and c are found from the real and imaginary components of z and y; and the experimental determination of constants is complete.

PROBLEMS

7-1. A wire table for standard annealed copper wire gives the following values: No. 14 AWG, diameter = 64.08 mils; resistance = 2.525 ohms per thousand feet. For what temperature (centigrade and Fahrenheit) is this table intended?

7-2. A wire table for pure hard-drawn copper wire gives the following values: No. 6 AWG, diameter = 0.1620 inch; resistance = 0.3731 ohm per thousand feet. For what temperature (centigrade and Fahrenheit) is this table intended?

7-3. Find the resistance of the wire of Prob. 7-1 at a temperature that is (*a*) 20°C higher than the temperature used for the table, and (*b*) 20°C lower. What are these temperatures on the Fahrenheit scale?

7-4. Find resistances (and temperatures) as required in Prob. 7-3 for the wire of Prob. 7-2.

7-5. Diameter of No. 10 AWG standard annealed copper wire is 101.9 mils; resistance is 0.9989 ohm per thousand feet at 20°C. Find resistance of the same gauge wire at the following temperatures:

a. 0°F (cold weather)
b. 0°C (freezing)
c. 20°C (room temperature)
d. 70°C (hot operating temperature)

for annealed, soft-drawn, and hard-drawn wire (12 values altogether) and give the percentage by which each differs from the wire-table value of 0.9989 ohm.

Note: It is helpful to remember that the resistance of No. 10 copper wire is 1 ohm per thousand feet.

7-6. Verify both of the d-c resistance values in Appendix 3 for ACSR of 900,000-circular-mil size. Is conductivity of the steel strands included in computing resistance? Why? Does the nominal size in circular mils include the steel strands?

7-7. Repeat Prob. 7-6 for 500,000-circular-mil ACSR.

7-8. Substitute Eq. 7-48 into Eq. 7-18 to prove that the Bessel function describes current distribution in a round wire.

7-9. Sketch six curves, each similar to Fig. 7-8 and together giving a series, as in Fig. 7-6, to show the distribution of current density within a round wire through a half cycle of time. Precision is not expected; it is enough to have the general shape of the curve correct.

7-10. Plot R_{dc}/R_{ac} as a function of a/δ (preferably using a logarithmic scale, as in Fig. 7-9). Draw three curves:

a. Exact, from Table 7-3.
b. Approximate, based on Eq. 7-58.
c. Approximate, based on Eq. 7-57.

Note the amount of error in the approximations.

7-11. Find R_{ac}/R_{dc}, the skin-effect resistance ratio, from Fig. 7-9, and also compute the value (correct to three significant figures) by any appropriate means. Find for round copper wire of the following sizes (AWG or diameter):

a. No. 0 at 1,000 cycles.
b. No. 10 at 1,000 cycles.
c. 1 inch at 60 cycles.
d. 0.5 inch at 60 cycles.
e. No. 10 at 1 megacycle.
f. No. 30 at 1,000 megacycles.

7-12. Repeat Prob. 7-11 for round copper wire of the following sizes (AWG):

a. No. 0 at 500 cycles.
b. No. 6 at 1,000 cycles.
c. No. 4/0 at 60 cycles.
d. No. 20 at 1 megacycle.
e. No. 40 at 1,000 megacycles.

7-13. Read skin-effect resistance ratio from Fig. 7-9 for aluminum wire, for sizes and frequencies listed in Prob. 7-11.

7-14. Read skin-effect resistance ratio from Fig. 7-9 for aluminum wire, for sizes and frequencies listed in Prob. 7-12.

7-15. It appears from Fig. 7-9 that l_{ac} and $1/r_{ac}$ for a given round wire are similar functions of frequency. Is there a corresponding similarity of l_{ac} and $1/r_{ac}$ for a semi-infinite block?

7-16. The following input impedance measurements were made at one end of a 40-mile telephone line, using a test frequency of 1,000 cycles:

With the far end open-circuited:

$$Z_{s(op)} = 250\underline{/-38.0^\circ}$$

With the far end short-circuited:

$$Z_{s(sh)} = 1{,}960\underline{/12.4^\circ}$$

Find Z_0, r, l, g, and c.

7-17. The following input impedance measurements were made at one end of a 6.3-mile telephone cable circuit, using a test frequency of 1,000 cycles:

With the far end open-circuited: $R = 177$, $X = -404$ ohms

With the far end short-circuited: $R = 458$, $X = -158$ ohms

Find Z_0, r, l, g, and c.

CHAPTER 8

ARTIFICIAL LINES

1. Purpose of Artificial Lines. When power transmission systems are being studied to determine the value of proposed new lines, new generators, or other additions to the plant, it is convenient to reproduce the system in miniature on a *network analyzer* or *calculating board.* It is not necessary to include all details of the actual system in the model, and one of the questions that must be decided in adapting the system to a calculating board is the best way to represent a transmission line.

The simplest representation of a transmission line would be by impedance only. If z, the impedance of the line per mile, were multiplied by the length of the line in miles, the resulting impedance Z could be used as a single lumped impedance on the calculating board to give a very approximate representation of the line.

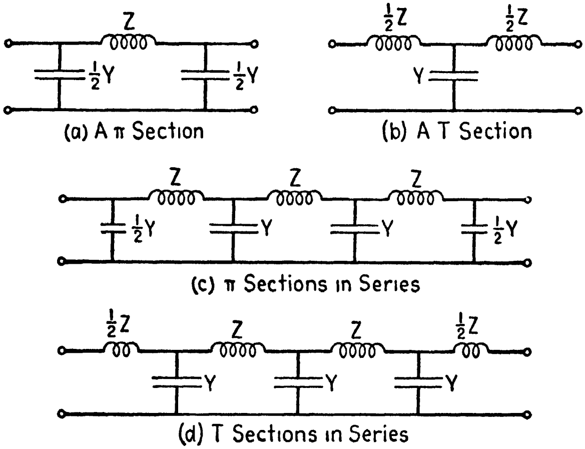

FIG. 8-1. Composition of artificial lines.

It is rarely satisfactory to neglect line capacitance completely. A better model is obtained if y, the admittance of the line per unit length, is multiplied by the length x to give a lumped admittance, $yx = Y$. Half of this lumped admittance, or $\frac{1}{2}Y$, can then be connected at each end of the lumped impedance on the calculating board. On a power line, the admittance is practically purely capacitive, and a condenser of the proper size is used on the board; the resulting *artificial line* is shown in Fig. 8-1*a*.

Artificial lines have a wide variety of uses, besides their application on calculating boards for power-system analysis. They are used in telephone work to duplicate the electrical behavior of a real line or cable circuit. They are used as pulse or time-delay circuits in radar and television systems. They are useful both in engineering practice and in theory.

2. Π and T Sections. Figure 8-1*a* shows the network that was described above to represent a transmission line on a calculating board. Because of its similarity to the letter Π it is called a Pi section. If the transmission-line constants are merely lumped in the artificial line, as proposed in the preceding discussion, the impedance[1] and admittance of the Π section are

$$Z = zx \tag{8-1}$$

$$Y = yx \tag{8-2}$$

where z is impedance per unit length of the real line
y is admittance per unit length of the real line
x is length of line represented

This simple lumping of constants gives what is called a *crude* Π *line*. It will be shown that this representation of a real line is satisfactory (although not exact) if the line to be represented is much shorter than a wavelength.

To represent a longer line, a number of Π sections may be connected in series, as in Fig. 8-1*c*. Such a series of sections is sometimes called a *ladder network*.

Instead of using an artificial line in the form of a Π section, the actual line can as well be represented by a T section as shown in Fig. 8-1*b*. The same degree of approximation is involved; the *crude* T *line*, like the crude Π line, is good only to represent a short real line.

T sections, like Π sections, can be connected in series to represent a line that is too long to be replaced by a single crude section. The result, shown in Fig. 8-1*d*, is identical with the ladder network resulting from Π sections in series, except at the ends.

If an artificial line is made of a number of crude Π or T sections, the more sections there are, the more nearly the artificial line resembles a real line. A 100-mile line is better represented by 100 sections that are equivalent to 1 mile each than by 2 sections equivalent to 50 miles each. One would expect (and, indeed, it turns out to be true) that if the number of sections used to represent a real line is increased without limit, the size of each section becoming smaller and smaller as the number is increased, the artificial line approaches an *exact* representation of the real line.

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[1] The figures of this chapter use symbols of inductance and capacitance to indicate the impedance and admittance elements. For greater precision of representation, resistances may be added in series with the inductances and shunted around the capacitances.

However, a real line and an artificial line are physically so different that an equivalence between the two is hardly to be taken for granted, even for very short lines. The equivalence of real lines and artificial lines is shown in the following paragraphs.

3. Open-circuit and Short-circuit Impedances. An artificial line is said to be equivalent to a real transmission line if the terminal behavior of one is identical with the terminal behavior of the other.

The parameters Z_0 and Bx (or γx) define the behavior of a transmission line, and two lines with the same Z_0 and Bx are electrically equivalent. It was shown in Chap. 7 that Z_0 and Bx can be determined from the open-circuit input impedance of a line by the following equations:

$$Z_0^2 = Z_{s(op)} \cdot Z_{s(sh)} \tag{8-3}$$

$$(\cos Bx)^2 = (\cosh \gamma x)^2 = \frac{Z_{s(op)}}{Z_{s(op)} - Z_{s(sh)}} \tag{8-4}$$

It follows that two lines with identical open-circuit and short-circuit impedances are electrically equivalent. This equivalence may apply at only one frequency, if open- and short-circuit impedances are identical at one frequency only, or the equivalence may be valid at all frequencies.

Similarly, an artificial line and a real line are equivalent if they have identical open-circuit and short-circuit impedances. If they behave the same when open-circuited, and when short-circuited, they will behave the same with all terminations.

Let us therefore find the open and short-circuit impedances of an artificial line, and then determine the parameters of a real line with the same open and short-circuit impedances. Such a real line will be equivalent to the artificial line. (It will be found that a real line and an artificial line are equivalent at one frequency only.)

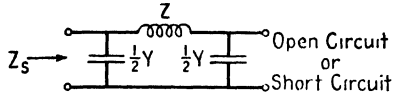


FIG. 8-2. A Π section for determining open and short-circuit impedances.

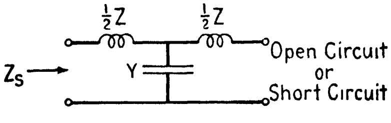


FIG. 8-3. A T section for determining open and short-circuit impedances.

$Z_{s(op)}$ and $Z_{s(sh)}$ of a Π or T network are readily computed. When introduced into Eqs. 8-3 and 8-4, they give the characteristic impedance and propagation constant of the equivalent transmission line. Thus the relation between a real line and an artificial line is established. The details of the computation follow.

Referring to Fig. 8-2, Z_s of a short-circuited Π network is the impedance of Z and $\frac{1}{2}Y$ in parallel. This gives

Symmetrical Π:
$$Z_{s(sh)} = \frac{Z \cdot \dfrac{2}{Y}}{Z + \dfrac{2}{Y}} = \frac{2Z}{ZY + 2} \tag{8-5}$$

The impedance looking into an open-circuited Π is

$$Z_{s(op)} = \frac{\left(Z + \dfrac{2}{Y}\right)\dfrac{2}{Y}}{Z + \dfrac{2}{Y} + \dfrac{2}{Y}} = \frac{2}{Y}\frac{ZY + 2}{ZY + 4} \tag{8-6}$$

Referring to Fig. 8-3, the impedances of a T section when short-circuited and open-circuited are

Symmetrical T:
$$Z_{s(sh)} = \frac{\dfrac{Z}{2} \cdot \dfrac{1}{Y}}{\dfrac{Z}{2} + \dfrac{1}{Y}} + \frac{Z}{2} = \frac{Z}{2}\frac{ZY + 4}{ZY + 2} \tag{8-7}$$

$$Z_{s(op)} = \frac{Z}{2} + \frac{1}{Y} = \frac{ZY + 2}{2Y} \tag{8-8}$$

4. Equivalent Π-section Network. Since the short-circuit and open-circuit impedances of a Π section are given by Eqs. 8-5 and 8-6, these may be used in Eq. 8-3 to find the characteristic impedance of an equivalent transmission line:

$$Z_0^2 = \frac{2}{Y}\frac{ZY + 2}{ZY + 4}\frac{2Z}{ZY + 2} = \frac{Z}{Y}\frac{4}{4 + ZY} = \frac{1}{\dfrac{Y}{Z} + \dfrac{Y^2}{4}} \tag{8-9}$$

It will be remembered that for a transmission line, $Z_0^2 = z/y$. For the artificial line of Eq. 8-9, $Z_0^2 = Z/Y$ times a correction factor, as shown in the third member of the equation. The correction is small if Z and Y are small.

Short-circuit and open-circuit impedances are used in Eq. 8-4 to find the propagation constant

$$(\cos Bx)^2 = (\cosh \gamma x)^2 = \frac{1}{1 - \dfrac{Z_{s(sh)}}{Z_{s(op)}}} = \frac{1}{1 - \dfrac{ZY(ZY + 4)}{(ZY + 2)^2}}$$
$$= \frac{(ZY + 2)^2}{4} \tag{8-10}$$

from which

$$\cos Bx = \cosh \gamma x = 1 + \tfrac{1}{2}ZY \tag{8-11}$$

As an alternative form that is sometimes more convenient, the trigonometric identity for the sine of half the angle can be introduced into Eq. 8-11 to give

$$\sin \tfrac{1}{2}Bx = \tfrac{1}{2}\sqrt{-ZY} \tag{8-12}$$

or

$$\sinh \tfrac{1}{2}\gamma x = \tfrac{1}{2}\sqrt{ZY} \tag{8-13}$$

Equations 8-9, 8-11, 8-12, and 8-13 are included in the summary of Table 8-1. They make it possible to find the parameters of a real transmission line, the impedance and admittance of the equivalent artificial-line section being known.[2]

The inverse problem is of at least equal importance; that is, given the parameters of a real line, what must be the impedance and admittance of an artificial line to be equivalent? The necessary information is, of course, contained in the above equations, but they must be solved for Z and Y explicitly.

Let us first solve for Z. We may write, using Eq. 8-10,

$$(\sin Bx)^2 = 1 - (\cos Bx)^2 = 1 - \frac{(ZY + 2)^2}{4} = -Z^2\left(\frac{Y^2}{4} + \frac{Y}{Z}\right) \tag{8-14}$$

Since this final parenthesis may be recognized, from Eq. 8-9, as the reciprocal of $Z_0{}^2$, it follows that

$$Z = jZ_0 \sin Bx \tag{8-15}$$

This is an explicit expression for Z, as desired, but it is usually more convenient to change its form as follows:

$$\begin{aligned} Z &= j\sqrt{\frac{z}{y}} \sin Bx = j\frac{z}{\sqrt{zy}} \sin Bx \\ &= \frac{zx}{-j\sqrt{zy}\,x} \sin Bx = zx\frac{\sin Bx}{Bx} \end{aligned} \tag{8-16}$$

To find Y, use Eq. 8-11 and substitute Eq. 8-15 for Z:

$$\begin{aligned} \frac{1}{2}Y &= \frac{1 - \cos Bx}{-Z} = \frac{1 - \cos Bx}{-jZ_0 \sin Bx} = \frac{1 - \cos Bx}{-j\sqrt{\frac{z}{y}} \sin Bx} \\ &= \frac{y}{-j\sqrt{zy}} \frac{1 - \cos Bx}{\sin Bx} = \frac{yx}{Bx} \tan \frac{1}{2} Bx \end{aligned} \tag{8-17}$$

---

[2] To find B or γ of the real line from Eq. 8-11, 8-12, or 8-13, it is necessary to evaluate the inverse sine or cosine (either circular or hyperbolic) of a complex quantity (see Sec. 6, Chap. 8).

The last transformation uses the relation $B = -j\sqrt{zy}$, and also the trigonometric identity of Formula 32 of Table II. The most convenient final form is

$$\frac{1}{2}Y = \frac{1}{2}yx\frac{\tan\frac{1}{2}Bx}{\frac{1}{2}Bx} \tag{8-18}$$

The equivalent expressions in hyperbolic functions are obtained if one writes, instead of Eq. 8-14, the following:

$$(\sinh\gamma x)^2 = (\cosh\gamma x)^2 - 1 = \frac{(ZY+2)^2}{4} - 1 = Z^2\left(\frac{Y^2}{4} + \frac{Y}{Z}\right) \tag{8-19}$$

The final parenthesis being $1/Z_0^2$, by Eq. 8-9, we have

$$Z = Z_0 \sinh\gamma x \tag{8-20}$$

Since $Z_0 = \sqrt{z/y} = zx/(\sqrt{zy}\,x)$, Eq. 8-20 may be written

$$Z = zx\frac{\sinh\gamma x}{\gamma x} \tag{8-21}$$

Y is then found from Eq. 8-11, using the hyperbolic form, and writing with the aid of Eq. 8-20

$$\frac{1}{2}Y = \frac{\cosh\gamma x - 1}{Z} = \frac{\cosh\gamma x - 1}{Z_0\sinh\gamma x} = \sqrt{\frac{y}{z}}\frac{\cosh\gamma x - 1}{\sinh\gamma x} \tag{8-22}$$

Using the identity $\tanh\frac{1}{2}x = (\cosh x - 1)/\sinh x$, this becomes

$$\frac{1}{2}Y = \frac{1}{2}yx\frac{\tanh\frac{1}{2}\gamma x}{\frac{1}{2}\gamma x} \tag{8-23}$$

This completes the derivation of the equations in Table 8-1 that refer to the Π section.

In all equations,

z is the impedance per unit length of the actual line
y is the admittance per unit length of the actual line
x is the length of the actual line
$B = -j\sqrt{zy}$ or $\gamma = \sqrt{zy}$
Z is the total series impedance of the section of artificial line
Y is the total shunt admittance of the section of artificial line

It is interesting to compare Eqs. 8-16 and 8-18 (or the hyperbolic forms 8-21 and 8-23) with the *crude* equations 8-1 and 8-2. Equation 8-1 says that $Z = zx$; this is approximate. Equation 8-16 gives the exact equivalence: $Z = zx\ (\sin Bx/Bx)$. The factor $\sin Bx/Bx$ is sometimes called a *correction factor.*

Similarly, Eq. 8-2 gives $Y = yx$. The exact value, given by Eq. 8-18, contains the *correction factor* $\tan \frac{1}{2}Bx/\frac{1}{2}Bx$. (These correction factors may also be given in hyperbolic functions, as in Eqs. 8-21 and 8-23.)

Two facts about the representation of a real line by an equivalent Π-section artificial line are now evident:

1. The approximation of Eqs. 8-1 and 8-2 is very good for a short line, as the correction factors ($\sin Bx/Bx$ and $\tan \frac{1}{2}Bx/\frac{1}{2}Bx$) both approach 1 for sufficiently small values of x. This justifies the statement that merely lumping the impedance and admittance of a real line to find the constants of an equivalent artificial line is satisfactory if the length of the real line is much less than a quarter-wavelength.

2. An artificial line is exactly equivalent to a real line at one frequency only. The correction factors of Eqs. 8-16 and 8-18 are functions of frequency (for B, which equals $-j\sqrt{zy}$, varies with frequency).

For a band of frequencies, an artificial line of one section can approximately represent a real line if the length of real line is much less than a quarter-wavelength at the highest frequency of the band. For so short a line, the correction factors are negligible at all frequencies under consideration. A longer real line can be represented by using a number of Π sections of artificial line in series, as in Fig. 8-1*c*. An extension of this argument justifies the statement that an infinite number of infinitesimally small sections of artificial line in series will exactly represent a real line of any length.

5. Equivalent T-section Network. The short-circuit and open-circuit impedances of a T section are given in Eqs. 8-7 and 8-8. To find the characteristic impedance of the equivalent real line, these are substituted into Eq. 8-3:

$$Z_0^2 = \frac{ZY+2}{2Y}\frac{Z}{2}\frac{ZY+4}{ZY+2} = \frac{Z}{Y}\frac{4+ZY}{4} = \frac{Z}{Y} + \frac{Z^2}{4} \qquad (8\text{-}24)$$

It will be noted that for the T section, as for the Π section, Z_0^2 differs from the *crude* value of Z/Y because of a correction factor that approaches unity for small values of Z and Y, and hence for short lines.

Using the short-circuit and open-circuit impedances of the T section in Eq. 8-4,

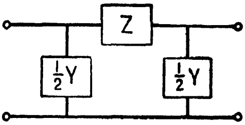
$$(\cos Bx)^2 = (\cosh \gamma x)^2 = \frac{(ZY+2)^2}{4} \qquad (8\text{-}25)$$

Since this turns out to be identical with Eq. 8-10 for the Π section, it follows that all equations deduced from Eq. 8-10 apply to a T section as well as to a Π section. Thus Eqs. 8-11, 8-12, and 8-13 all apply to a T section, and are so classified in Table 8-1.

TABLE 8-1
ARTIFICIAL-LINE FORMULAS

Π *Section*

Line to Π *Section:*



$$Z = zx\,\frac{\sin Bx}{Bx} = zx\,\frac{\sinh \gamma x}{\gamma x}$$

$$\frac{1}{2}\,Y = \frac{1}{2}\,yx\,\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} = \frac{1}{2}\,yx\,\frac{\tanh \frac{1}{2}\gamma x}{\frac{1}{2}\gamma x}$$

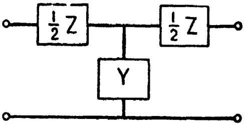
Π *Section to Line:*

$$\left.\begin{array}{ll} \cos Bx = 1 + \frac{1}{2}ZY & \cosh \gamma x = 1 + \frac{1}{2}ZY \\ \sin \frac{1}{2}Bx = \frac{1}{2}\sqrt{-ZY} & \sinh \frac{1}{2}\gamma x = \frac{1}{2}\sqrt{ZY} \end{array}\right\}\text{ alternatives}$$

$$Z_0 = \frac{1}{\sqrt{\dfrac{Y}{Z} + \dfrac{Y^2}{4}}}$$

T *Section*

Line to T *Section:*



$$\frac{1}{2}\,Z = \frac{1}{2}\,zx\,\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} = \frac{1}{2}\,zx\,\frac{\tanh \frac{1}{2}\gamma x}{\frac{1}{2}\gamma x}$$

$$Y = yx\,\frac{\sin Bx}{Bx} = yx\,\frac{\sinh \gamma x}{\gamma x}$$

T *Section to Line:*

$$\left.\begin{array}{ll} \cos Bx = 1 + \frac{1}{2}ZY & \cosh \gamma x = 1 + \frac{1}{2}ZY \\ \sin \frac{1}{2}Bx = \frac{1}{2}\sqrt{-ZY} & \sinh \frac{1}{2}\gamma x = \frac{1}{2}\sqrt{ZY} \end{array}\right\}\begin{array}{l}\text{alternatives} \\ \text{(}\textit{Note:}\text{ Same as} \\ \text{for } \Pi \text{ section.)}\end{array}$$

$$Z_0 = \sqrt{\frac{Z}{Y} + \frac{Z^2}{4}}$$

Real Line:

$Z_0 = \sqrt{z/y}$ $\quad B = -j\sqrt{zy}$ $\quad \gamma = jB = \sqrt{zy}$ $\quad x$ = length of line

To find explicit expressions for Z and Y of a T section to represent a real line, use Eq. 8-25 to write

$$(\sin Bx)^2 = 1 - (\cos Bx)^2 = -Y^2\left(\frac{Z^2}{4} + \frac{Z}{Y}\right) \qquad (8\text{-}26)$$

Introducing Z_0^2 from Eq. 8-24,

$$(\sin Bx)^2 = -Y^2Z_0^2 \qquad (8\text{-}27)$$

from which

$$Y = \frac{j}{Z_0}\sin Bx \qquad (8\text{-}28)$$

By taking steps parallel to those of Eq. 8-16, this gives

$$Y = yx\frac{\sin Bx}{Bx} \qquad (8\text{-}29)$$

The hyperbolic-function form, similarly derived, is

$$Y = yx\frac{\sinh \gamma x}{\gamma x} \qquad (8\text{-}30)$$

To find Z for the T section, substitute Eq. 8-28 for Y in Eq. 8-11:

$$\frac{1}{2}Z = \frac{1 - \cos Bx}{-(j/Z_0)\sin Bx} = \frac{1}{2}zx\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} \qquad (8\text{-}31)$$

The detailed steps in this derivation are the same as those of Eq. 8-17. If the hyperbolic forms of the expressions are used, instead of the circular, the result is similar:

$$\frac{1}{2}Z = \frac{1}{2}zx\frac{\tanh \frac{1}{2}\gamma x}{\frac{1}{2}\gamma x} \qquad (8\text{-}32)$$

The discussion in Sec. 5, following Eq. 8-23, may be applied to T sections as well as to Π sections. That is, the correction factors that appear in the exact equations for artificial lines, but not in the *crude* equations, approach unity for short sections of line; and the equivalence of a real and an artificial line is exact at one frequency only.

6. Evaluation of Inverse Functions. Equations 8-11 to 8-13 are for the purpose of finding Bx or γx of a real line. From Eq. 8-11, for instance,

$$\cos Bx = 1 + \tfrac{1}{2}ZY \qquad (8\text{-}33)$$

$$Bx = \cos^{-1}(1 + \tfrac{1}{2}ZY) \qquad (8\text{-}34)$$

To find Bx, knowing Z and Y, it is necessary to evaluate the inverse cosine of a complex quantity.[3]

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[3] The objection to the notation used in Eq. 8-34 for the inverse cosine is recognized, but the notation is familiar to most readers and rarely leads to actual mis-

This can be done in various ways. A few are listed in order of difficulty:

1. The simplest evaluation of the *inverse cosine* is an approximation that is good if the imaginary component of the cosine is small. Let us write, as a pure mathematical expression with (at present) no physical significance,

$$\cos(m + jn) = x + jy \tag{8-35}$$

or

$$m + jn = \cos^{-1}(x + jy) \tag{8-36}$$

where m, n, x, and y are any real numbers. It is assumed that x and y are known, and m and n are to be found (as in Eq. 8-34). The *approximate* formulas are[4]

$$m \cong \cos^{-1} x \tag{8-37}$$

$$n \cong -\frac{y}{\sin m} \tag{8-38}$$

---

understanding. The alternative form *arc cos* (for *cos*$^{-1}$) implies angular measure so strongly that it is not desirable. The more logical *inv cos* is not familiar to everyone, especially in the United States. To avoid any possible confusion, powers of the trigonometric functions are written as in Eqs. 8-25 and 8-26.

[4] *Derivation*: These approximate forms can be derived from Formula 49 of Table II, but they are more readily seen as follows. Expand Eq. 8-35 by Formula 25 of Table II

$$\begin{aligned}\cos(m + jn) &= \cos m \cosh n - j \sin m \sinh n \\ &= x + jy \end{aligned}\tag{8-42}$$

Equating real and imaginary components,

$$x = \cos m \cosh n \tag{8-43}$$

$$y = -\sin m \sinh n \tag{8-44}$$

If n is so small that $\cosh n$ is very nearly equal to 1,

$$x \cong \cos m \qquad \text{and} \qquad m \cong \cos^{-1} x \tag{8-45}$$

If n is so small that $\sinh n$ is very nearly equal to n,

$$y \cong -n \sin m \qquad \text{or} \qquad n \cong -\frac{y}{\sin m} \tag{8-46}$$

These are the approximate formulas for the inverse cosine, and the approximate formulas for the inverse sine are derived in a similar manner.

The requirement for good approximation is that the true value of v for the inverse sine, and of n for the inverse cosine, must be much smaller than 1.

These are good approximations if n is small; for slide-rule accuracy, n should be less than 0.1. This is often true in the computations associated with low-loss lines.

The corresponding *approximate* expression for the *inverse sine* is as follows. If

$$\sin(u + jv) = x + jy$$

or

$$u + jv = \sin^{-1}(x + jy) \tag{8-39}$$

where u, v, x, and y are real numbers, *approximately*

$$u \cong \sin^{-1} x \tag{8-40}$$

$$v \cong \frac{y}{\cos x} \tag{8-41}$$

The approximation is good if v is small; for slide-rule accuracy, v should be less than 0.1.

Since x and y are real, m and n, or u and v, are readily found by the use of Table III or any other table of real sines and cosines.

2. If the above approximations are not applicable, the exact formulas of Table II may be used. Formula 49 gives the inverse sine, and Formulas 49 and 51 together give the inverse cosine.

Formula 53 gives the inverse hyperbolic sine. Formula 54 gives the inverse hyperbolic cosine. Formula 53 can be transformed to resemble Formula 49 if desired, and there are many other alternative forms of expressions for the inverse functions.[5]

3. If the accuracy of values read from a chart is sufficient, the chart of Fig. 4-5 or any equivalent chart of the complex sine and cosine functions can be used, working backward to determine the inverse sine or cosine.[6] Such a chart may be used for either circular or hyperbolic functions by proper interpretation of the coordinates. Nomograms are also available.[7]

4. Infinite series are also available for evaluating the inverse functions. Being rather slowly convergent, these are tedious to use, specially with complex quantities, and the formulas to which we have referred are usually preferable.

7. Frequency Response. It has been emphasized that an artificial line gives a true representation of a particular real line at one frequency only.

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[5] A number of useful formulas for inverse complex hyperbolic functions, both exact and approximate, are included in J. Rybner, "Nomograms of Complex Hyperbolic Functions" (Ref. 26).

[6] The most accurate and extensive charts are by Kennelly (Ref. 12).

[7] J. Rybner (Ref. 26).

The shorter the section of real line represented by the artificial line, the less variation there is with frequency; and if an artificial line is to be required to give fair equivalence to a real line for a wide band of frequencies, the artificial line should be made of many small sections in series.

Attempts are often made to study transient phenomena such as traveling waves on artificial transmission lines. If this is done, the results have to be interpreted with great care. Any transient disturbance can be analyzed (as a Fourier integral, or by Laplace transform methods) into many sinusoidal components distributed through the frequency spectrum. Since an artificial line will not react in the same way as a real line to the various frequency components, it cannot give a correct response to a transient disturbance. The artificial line may actually block the passage of higher frequency components that the real line would transmit freely; an artificial line acts as a low-pass filter, as will be discussed in the next two chapters.

To make an artificial line more applicable to transient studies, an artificial line "with distributed constants" has been devised. This consists of a number of coils of wire with tin foil interleaved between layers of the coils. These coils provide not only inductance and resistance but capacitance as well; capacitance of the line is simulated by capacitance from the wire of the coil to the tin foil. Thus the capacitance is not inserted in lumps, but is distributed along the entire length of the wire of which each coil is wound.

There is some advantage in an arrangement of this kind for studying transients. However, the equivalence to a real line is not complete. There are several reasons why it is not, the most obvious being that there is mutual inductance between the various turns of wire that comprise a coil of the artificial line, whereas such mutual inductance does not exist on a real transmission line.

Despite the inexact response, artificial lines are often useful for propagating pulses or other transient signals. An interesting application is to provide a time delay, as in certain radar circuits. A pulse or other signal can be delayed without being changed in shape by sending it along a transmission line of proper length. An actual transmission line is often not practicable because it would have to be too long. An artificial line can sometimes be used to produce the desired delay.

In some systems, a tube of mercury is used for a delay line; the signal is changed from an electric wave to a mechanical vibration, and as such it is transmitted along the mercury column. The mass of the mercury is analogous to inductance, the mechanical elasticity of the mercury and the containing tube to capacitance. Dimensions are such that the desired time delay can be obtained with a tube of reasonable length. Such a mercury tube is not, of course, an artificial line. It is a real transmission line with distributed constants, propagating mechanical instead of electrical waves.

PROBLEMS

8-1. Design a Π-section artificial line equivalent at 1,000 cycles to 10 miles of the telephone line of Fig. 5-2.

8-2. Design a T-section artificial line equivalent at 1,000 cycles to 10 miles of the telephone line of Fig. 5-2.

8-3. Revise the artificial line of Prob. 8-1 to give equivalence at 3,000 cycles.

8-4. Revise the artificial line of Prob. 8-2 to give equivalence at 3,000 cycles.

8-5. Design a Π-section artificial line equivalent at 60 cycles to 300 miles of the power line of Fig. 5-3. (The artificial line represents one phase to neutral.)

8-6. Design a T-section artificial line equivalent at 60 cycles to 300 miles of the power line of Fig. 5-3. (The artificial line represents one phase to neutral.)

8-7. Design a Π-section artificial line equivalent at 1,000 cycles to 10 miles of the telephone cable circuit of Fig. 5-5.

8-8. Design a T-section artificial line equivalent at 1,000 cycles to 10 miles of the telephone cable circuit of Fig. 5-5.

8-9. Discuss the possibility of designing an artificial line equivalent at 3,000 cycles to 10 miles of the telephone cable circuit of Fig. 5-5, using (*a*) 1 section, (*b*) 2 sections, and (*c*) 3 or more sections. Would Π or T sections be better?

8-10. Z in the Π section of Fig. 8-1*a* is $104.4 + j230$ and Y is $j526 \times 10^{-6}$ at 1,000 cycles (a *crude* representation of the line of Prob. 8-1). Find β, α, and Z_0 for the line that is exactly equivalent to this Π.

8-11. Z in the T section of Fig. 8-1*b* is $104.4 + j230$ and Y is $j526 \times 10^{-6}$ at 1,000 cycles (a *crude* representation of the line of Prob. 8-1). Find β, α, and Z_0 for the line that is exactly equivalent to this T.

8-12. Repeat Prob. 8-10 for $Z = 104.4 + j690$ and $Y = j1{,}578$ (in a Π section) at 3,000 cycles.

8-13. Repeat Prob. 8-11 for $Z = 104.4 + j690$ and $Y = j1{,}578$ (in a T section) at 3,000 cycles.

8-14. A Π section has $Z = 10.44 + j\omega(0.0366)$ and $Y = j\omega(8.38)10^{-8}$. Compute and plot Z_0 (magnitude and angle), α, and velocity of propagation for an equivalent line as functions of frequency (from 1,000 to 6,000 cycles).

8-15. Repeat Prob. 8-14 for a T section.

CHAPTER 9

APPLICATION OF TRANSMISSION-LINE THEORY TO TELEPHONE LINES

1. Requirements of Telephone Service. Telephone lines are designed primarily to transmit speech, although they serve various other purposes. Music, picture transmission, and telegraph signals are commonly carried on telephone lines. Speech may be transmitted either by sending currents of the actual voice frequencies or by using the voice to modulate a current of higher frequency in carrier-current transmission.

For a signal that transmits sound, whether the message is speech or music or other audible vibrations, ideal transmission would permit equally good reception of all frequencies from 20 to 20,000 cycles per second. This range of frequencies can be detected by better than average ears. Ideal transmission, however, is quite unnecessary. Even high-quality music can be reproduced satisfactorily by using less than half the ideal range, and there is little loss of fidelity if only frequencies below 8,000 cycles are reproduced. For transmitting speech, most of the information is contained in the frequency range between 500 and 2,000 cycles; a telephone system that delivers only this range is quite practical although the listener is conscious that the tone is distorted. The following list gives a general idea of the frequency ranges used for telephone work:

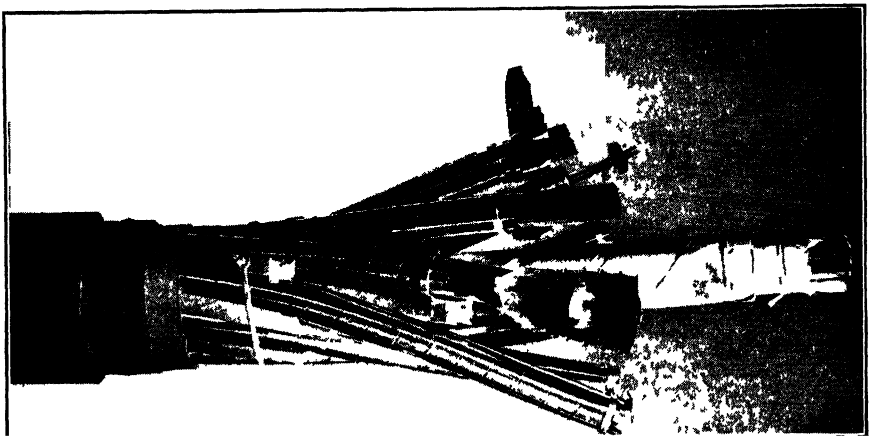
20–20,000 cycles	ideal frequency range
500– 2,000 cycles	good enough for speech
250– 2,750 cycles	present general average for speech transmission
100– 3,500 cycles	best present practice, speech
100– 5,000 cycles	program circuits, music
35– 8,000 cycles	special program circuits

When carrier-current transmission is used, the audio-frequency signal modulates a much higher *carrier* frequency. The carrier frequency is something between 6 and 140 kilocycles per second on most of the long telephone lines of the United States. Carrier frequencies up to three thousand kilocycles per second are used for long-distance telephone service in coaxial cable. (Coaxial cable is also used to transmit television signals, which require frequencies up to several thousand kilocycles.)

Modulation produces sum and difference frequencies on either side of the carrier frequency. One of these *side bands* can be suppressed without losing quality of transmission, and this is usually done. The carrier frequency can also be filtered out if it is reintroduced at the receiving end. This leaves, as the minimum to be transmitted in one carrier channel, a band of frequencies equal to the frequency range of the modulating signal. The carrier band may be either just above or just below the carrier frequency.

A pair of wires may carry a dozen or more telephone messages at the same time in different carrier channels. Each message is separated from the others at the receiving end of the line by a filter that passes the frequency band of a single channel only.

As an example, the lowest carrier frequency of the Type J carrier system is 36 kilocycles; if this carrier is modulated with an audio-frequency signal having components to 3 kilocycles, and the lower side band is suppressed, that particular channel will be required to transmit frequencies from 36 to 39 kilocycles. The next channel of this system has a 40-kilocycle carrier, and must transmit to 43 kilocycles. Twelve such channels, between 36 and 83 kilocycles, constitute half of the Type J system; twelve other channels between 93 and 140 kilocycles carry messages in the opposite direction. Thus twelve conversations are carried by this system on one pair of wires.



A cable of eight ⅜-inch coaxial lines for telephone or video transmission. The cable also contains a number of auxiliary wire circuits, in the center core and between coaxial lines, for alarm, control, etc. Inner wires of coaxial lines are held by polyethylene discs an inch apart. Each coaxial line is wrapped with copper and permalloy tapes for shielding; heavy paper and a lead sheath give protection. (*Courtesy of Bell Telephone Laboratories.*)

Whether voice-frequency or carrier-current transmission is used, the signal at the receiving end of a telephone line must be strong enough to be audible, and its wave form must be enough like the wave form at the sending end of the line to make it intelligible. To provide this necessary condition, attenuation and distortion must be avoided or at least controlled.

2. Distortion. Attenuation and distortion are not entirely distinct. One type of distortion is merely unequal attenutation of different frequencies. Another kind of distortion results from propagating different frequencies at different speeds.

Any signal to be transmitted can be analyzed into component frequencies by Fourier analysis. If, at the receiving end, all frequencies are in the correct proportion to each other, and all are in the proper relative phase relation, they will combine to give a signal identical (except in size) to the transmitted signal.

If the amplitudes of the various frequency components at the receiving end are not in the correct proportion, *frequency distortion* results.[1] As the name implies, different frequencies are propagated with different degrees of attenuation. If high notes are received and low notes are lost, the result is "tinny," as in the early telephones, old phonographs, and some cheap radios. Speech so distorted is perfectly intelligible unless the distortion is extreme, but sounds nasal.

If, on the contrary, the low notes are received and high frequencies are suppressed, the result is "booming," like the sound from a radio in another room. Speech is hard to understand without the high frequencies, for the high-frequency components, occurring in the consonant sounds, give shape to the words.

The ear is not sensitive to phase relation between the frequency components of a sound, and if a fundamental note and its overtones are reproduced with an incorrect phase relation it is not noticeable. With picture transmission, however, including television, and in the transmission of pulses of a given shape, correct phase relation is vitally necessary. If, for example, it were desired to transmit a picture involving alternate stripes of black and white as in a checkerboard pattern, or the picture of a picket fence, the fundamental and its third harmonic would be related as in Fig. 9-1*a*. Should phase distortion shift the third harmonic relative to its fundamental, the received signal could be as in *b* of that figure.

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[1] This type of distortion is sometimes called amplitude distortion, which is reasonable as it is the amplitudes that are distorted. However, because of the danger of confusion with *nonlinear distortion,* which is also called amplitude distortion, the term *frequency distortion* has been adopted. (*Nonlinear distortion,* important in amplifiers, does not occur in transmission lines comprising all linear elements.) *Delay distortion,* or phase distortion, is the other type of distortion to be distinguished.

With all harmonics shifted, a triangular wave might appear where a flat-top wave was wanted, with unfortunate results for the picture.

This type of distortion occurs if one frequency is delayed more or less than another; it is called *delay distortion.* Since it gives incorrect relative phase, it is also called phase distortion. It occurs on a transmission line if the velocity of propagation is greater for one frequency than for another. On typical telephone lines, as may be seen from Fig. 5-2, low-frequency waves have lower velocity. This is much more marked with telephone cables, as in Fig. 5-5. (A *loaded* cable, however, as will be considered, transmits higher frequencies at lower velocity.)

Although a shift of relative phase of frequency components is not important in sound transmission, the delay must not be so great that different components of the same sound arrive at noticeably different times. The click of a typewriter key, for instance, comprises many frequencies, and if the high-frequency components reach the ear of the listener as much as a few hundredths of a second earlier than the low components, the result is quite unlike the click of a typewriter key. Many musical instruments are badly reproduced if there is too much delay distortion. The working rule of the telephone system is to keep the delay time of all frequencies on program circuits within 10 milliseconds of the 1,000-cycle delay time. This is done by using *time-delay equalizers* which are networks in series with the line; these networks produce a delay similar to the delay caused by the line, but they delay most the frequencies that are delayed least in traversing the line, thereby equalizing the over-all delay.

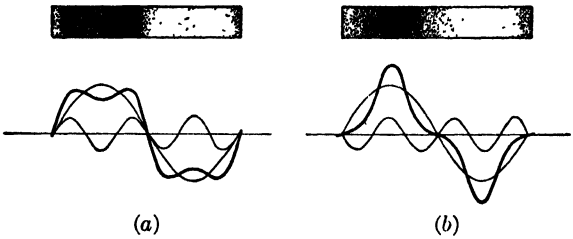
(*a*) (*b*)

FIG. 9-1. Example of delay distortion.

Speech circuits do not demand careful equalizing of delay, but picture-transmission circuits require great care.

A related problem is "echo". If part of an east-west signal is reflected at the terminal equipment and is sent back into the west-east channel, it will return to the speaker at the eastern end of the line, and he will hear his own words after a short delay. This is disturbing, and the greater the delay, the more annoying the echo. To reduce the effect of echo, high velocity is desirable.

3. Termination in Characteristic Impedance. It is standard practice to terminate telephone lines in an impedance equal, or approximately equal, to the characteristic impedance of the line. Several reasons for such termination may be listed:

1. To avoid reflections on a two-way two-wire line with repeaters (amplifiers).
2. To minimize reflection crosstalk on carrier lines.
3. To provide maximum-power transfer.
4. To provide proper circuit impedance (usually 600 ohms) for correct functioning of terminal equipment.
5. To avoid frequency distortion resulting from resonance of the line.

Of these, 3 and 5 will be discussed below. Point 1 is explained in Sec. 13 of this chapter. (Points 2 and 4 are outside the scope of this book.)

If the impedance at the receiving end of a line differs from the characteristic impedance, certain frequencies are intensified or diminished by *resonance.* For instance, if the terminating impedance is too high, frequency components for which the length of the line is approximately an odd multiple of a quarter-wavelength are increased, and frequency distortion results.

If the terminating impedance is equal to Z_0, resonance effects are eliminated. There is an incident wave train, but no reflected wave; consequently there are no standing waves and no maxima or minima of current or voltage, but only a gradual and uniform attenuation from transmitter to receiver.[2]

Another advantage of the line terminated in its characteristic impedance is that it provides *maximum-power transfer* from line to load. The importance of this in obtaining a useful signal is obvious.

The mathematical treatment of a line terminated in its characteristic impedance is exceptionally simple. Since there is only an incident wave train, the voltage and current are related at every point by the characteristic impedance:

$$\frac{V}{I} = Z_0 \tag{9-1}$$

Both voltage and current change phase along the line by an amount (βx) proportional to distance. They become smaller from transmitter to receiver at an exponential rate, the exponent (αx) being proportional to distance. From equations of Chap. 1 (1-41 and 1-43) for voltage and current,

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[2] If a telephone line is much shorter than a wavelength, it is possible to disregard resonance, and to terminate the line as is convenient for other reasons. For instance, a line 4 miles long is about $\frac{1}{10}$ wavelength at a frequency of 5,000 cycles per second. A glance at the chart for cos Bx (Fig. 4-2) shows that resonant effects in a line of this length, although appreciable, are small. For most purposes, it is good enough to consider the resistance and reactance of so short a line as lumped, without using long-line theory.

$$V_s = V_r e^{\alpha x} e^{j\beta x} \tag{9-2}$$

$$I_s = I_r e^{\alpha x} e^{j\beta x} \tag{9-3}$$

4. Attenuation. In each of these equations, the first exponential has to do with magnitude and the second with phase angle only. The ratio of magnitudes is given by

$$\left| \frac{V_s}{V_r} \right| = e^{\alpha x} \qquad \left| \frac{I_s}{I_r} \right| = e^{\alpha x} \tag{9-4}$$

The phase angle between voltage and current is everywhere the angle of the characteristic impedance. Input power is $|V_s|\,|I_s|\cos$ (*angle of* Z_0), and power delivered by the line to the load is $|V_r|\,|I_r|\cos$ (*angle of* Z_0). The ratio of sending-end power to receiving-end power is therefore

$$\frac{P_s}{P_r} = \frac{|V_s|\,|I_s|}{|V_r|\,|I_r|} = e^{2\alpha x} \tag{9-5}$$

Taking the natural logarithm of each side,

$$\alpha x = \frac{1}{2} \ln \frac{P_s}{P_r} \tag{9-6}$$

This is an expression for attenuation on a line that is terminated in its characteristic impedance in terms of the natural (Naperian) logarithm of the ratio of output to input power, and αx, although a pure number, is usually expressed as so-many *nepers*.

Decibels. In the United States, the common unit of attenuation is the *decibel*. (Nepers are more frequently used in Europe.) The decibel is a unit derived from the *bel*. If attenuation is 1 bel, the output power is one-tenth of the input power. If the attenuation is 2 bels (as when there are two attenuating networks of 1 bel each, in series) the output power is $\frac{1}{10}$ of $\frac{1}{10}$, or $\frac{1}{100}$, of the input power. The general definition is

$$\text{Attenuation measured in bels} = \log_{10} \frac{P_s}{P_r} \tag{9-7}$$

The bel is rather too large a unit for practical use, so the decibel is used. The decibel is $\frac{1}{10}$ bel, being defined by

$$\text{Attenuation measured in decibels} = 10 \log_{10} \frac{P_s}{P_r} \tag{9-8}$$

It is apparent from the similarity of Eqs. 9-6 and 9-8 that there is a relation between attenuation in decibels and attenuation in nepers. Taking the common logarithm of each side of Eq. 9-5,

$$\log \frac{P_s}{P_r} = \log e^{2\alpha x} = 2\alpha x \log e = 0.8686\alpha x \qquad (9\text{-}9)$$

Multiplying by 10 to get decibels (abbreviated db),

$$\text{Attenuation in decibels} = 10 \log \frac{P_s}{P_r} = 8.686\alpha x \qquad (9\text{-}10)$$

From this, it is seen that attenuation in decibels can be found by multiplying attenuation in nepers by 8.686.

The decibel unit, in general, is a means of comparing two values of power. As we have seen, the attenuation loss of a transmission line can be expressed in decibels. So, also, can the loss of any electrical network between input and output terminals. If the network should be an amplifier instead of a dissipative network, the loss will be negative (the logarithm of a ratio less than 1 is negative), and the amplifier is said to have a *gain* of so-many decibels.

The great advantage of a logarithmic unit such as the decibel (or the neper) is that the total loss of a number of transmission lines and other networks in series is merely the sum of the losses of the individual units. The gain of amplifiers can also be included in such a sum. Thus, if three sections of telephone line have attenuations of 20 db, 15 db, and 23 db, respectively, and there are four amplifiers in the system, each having 12 db gain, the net loss is 10 db, indicating that the output power is one-tenth of the input power. A power-level diagram of such a system of lines and amplifiers, as in Fig. 9-2, is used in telephone work. The diagram shows at a glance that the power level at its lowest point is 22 db below the input power, and at its highest point it is up 12 db.

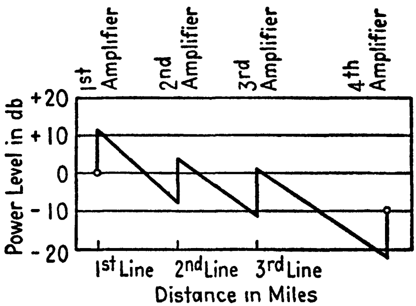

FIG. 9-2. Power-level diagram.

In this illustration, power at any point on the line is compared with input power. It is often more convenient to compare power at any point with some arbitrary power value that is generally accepted as a standard. For example, power level on the telephone systems of the United States is commonly expressed in decibels referred to 1 milliwatt as standard. This telephone standard is sometimes distinguished by the symbol dbm. Thus, a power level of 20 dbm means 100 milliwatts, and a power level of -10 dbm means 0.1 milliwatt. One milliwatt is called *zero dbm.* If Fig. 9-2 is interpreted with reference to this telephone standard, input to the first amplifier is seen to be 1 milliwatt and output of the last amplifier is 0.1 milliwatt. (Other standards are used for 0 db for other purposes. Acoustics, for instance, uses a different standard.)

Table 9-1 gives power ratios corresponding to decibel values. It will be found convenient to remember a few of these. The whole system is based on the relation:

10 db gives a power ratio of 10

from which

20 db gives a power ratio of 100
30 db gives a power ratio of 1,000
. . . . .
60 db gives a power ratio of 1,000,000

For smaller differences:

1 db gives a power ratio of about $\frac{5}{4}$
3 db gives a power ratio of about 2
7 db gives a power ratio of about 5

If only these are remembered, other values can be estimated as needed by adding decibels and multiplying power ratios. Thus 4 db equals 3 + 1 db, giving a power ratio of $2 \times \frac{5}{4}$ or $\frac{5}{2}$.

Voltage and Current Ratios. If the receiving-end impedance of a telephone line is the characteristic impedance Z_0, it follows that the input impedance to the line is also Z_0. Then

$$V_s = I_s Z_0 \qquad V_r = I_r Z_0 \tag{9-11}$$

and we can write, as in Eq. 9-5,

$$\frac{P_s}{P_r} = \frac{|I_s Z_0|}{|I_r Z_0|}\frac{|I_s|}{|I_r|} = \frac{|I_s|^2}{|I_r|^2} \tag{9-12}$$

or, if preferred,

$$\frac{P_s}{P_r} = \frac{|V_s|}{|V_r|}\frac{|V_s/Z_0|}{|V_r/Z_0|} = \frac{|V_s|^2}{|V_r|^2} \tag{9-13}$$

These express attenuation in terms of voltage ratio or current ratio *for a line terminated in its characteristic impedance.* By Eq. 9-8,

$$\text{Attenuation in db} = 10 \log_{10} \frac{|V_s|^2}{|V_r|^2} = 20 \log \frac{|V_s|}{|V_r|} \tag{9-14}$$

$$= 20 \log \frac{|I_s|}{|I_r|} \tag{9-15}$$

TABLE 9-1
POWER RATIO IN DECIBELS*

Decibels	Power ratio		Current or voltage ratio		Decibels	Power ratio		Current or voltage ratio	
	Gain	Loss	Gain	Loss		Gain	Loss	Gain	Loss
0.1	1.02	0.977	1.01	0.989	4.2	2.63	0.380	1.62	0.617
0.2	1.05	0.955	1.02	0.977	4.4	2.75	0.363	1.66	0.603
0.3	1.07	0.933	1.03	0.966	4.6	2.88	0.347	1.70	0.589
0.4	1.10	0.912	1.05	0.955	4.8	3.02	0.331	1.74	0.575
0.5	1.12	0.891	1.06	0.944	5.0	3.16	0.316	1.78	0.562
0.6	1.15	0.871	1.07	0.933	5.5	3.55	0.282	1.88	0.531
0.7	1.17	0.851	1.08	0.923	6.0	3.98	0.251	1.99	0.501
0.8	1.20	0.832	1.10	0.912	6.5	4.47	0.224	2.11	0.473
0.9	1.23	0.813	1.11	0.902	7.0	5.01	0.199	2.24	0.447
1.0	1.26	0.794	1.12	0.891	7.5	5.62	0.178	2.37	0.422
1.1	1.29	0.776	1.13	0.881	8.0	6.31	0.158	2.51	0.398
1.2	1.32	0.759	1.15	0.871	8.5	7.08	0.141	2.66	0.376
1.3	1.35	0.741	1.16	0.861	9.0	7.94	0.126	2.82	0.355
1.4	1.38	0.724	1.17	0.851	9.5	8.91	0.112	2.98	0.335
1.5	1.41	0.708	1.19	0.841	10.0	10.00	0.100	3.16	0.316
1.6	1.44	0.692	1.20	0.832	12.0	15.8	0.063	3.98	0.251
1.7	1.48	0.676	1.22	0.822	14.0	25.1	0.040	5.01	0.199
1.8	1.51	0.661	1.23	0.813	16.0	39.8	0.025	6.31	0.158
1.9	1.55	0.646	1.24	0.803	18.0	63.1	0.016	7.94	0.126
2.0	1.58	0.631	1.26	0.794	20.0	100.0	0.010	10.00	0.100
2.2	1.66	0.603	1.29	0.776	25.0	3.16×10^{2}	3.16×10^{-3}	17.8	0.056
2.4	1.74	0.575	1.32	0 759	30.0	10^{3}	10^{-3}	31.6	0.032
2.6	1.82	0.550	1.35	0.741	35.0	3.16×10^{3}	3.16×10^{-4}	56.2	0.018
2.8	1.91	0.525	1.38	0.724	40.0	10^{4}	10^{-4}	100.0	0.010
3.0	1.99	0.501	1.41	0.708	45.0	3.16×10^{4}	3.16×10^{-5}	177.8	0.006
3.2	2.09	0.479	1.44	0.692	50.0	10^{5}	10^{-5}	316	0.003
3.4	2.19	0.457	1.48	0.676	55.0	3.16×10^{5}	3.16×10^{-6}	562	0.002
3.6	2.29	0.436	1.51	0.661	60.0	10^{6}	10^{-6}	1,000	0.001
3.8	2.40	0.417	1.55	0.646	65.0	3.16×10^{6}	3.16×10^{-7}	1,780	0.0006
4.0	2.51	0.398	1.58	0.631	70.0	10^{7}	10^{-7}	3,160	0.0003

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* From F. E. Terman, "Radio Engineers' Handbook," McGraw-Hill Book Company, Inc., New York, 1943, and "Radio Engineering," 3d ed., McGraw-Hill Book Company, Inc., New York, 1947.

It must be emphasized most strongly that Eqs. 9-14 and 9-15, being derived from Eqs. 9-12 and 9-13, are valid only to compare *voltages or currents feeding into equal impedances,* for the decibel is fundamentally a means of comparing power.

5. Maximum Power Transfer. It was mentioned that termination of a line in its characteristic impedance not only avoids resonance, but also provides maximum power transfer. This statement is exact if the characteristic impedance is pure resistance; otherwise it is an approximation, but for most practical operation of telephone lines it is quite a good approximation.

The well-known maximum-power-transfer theorem, referring to a network of fixed impedance and constant internal voltage, joined at a pair of terminals to another network of variable impedance, says that *maximum power is absorbed by the receiving network when the impedance looking into the receiving network at the junction is the conjugate of the impedance looking back into the generating network* (both impedances being measured with the receiving network disconnected and the voltage source or sources inactive).

If Z_b of Fig. 9-3 can be varied, and E is a source of voltage without impedance, the theorem[3] tells us that Z_b will receive maximum power when Z_b is the conjugate of Z_a; that is, when the resistance of Z_b equals the resistance of Z_a and the reactance of Z_b is equal but opposite to the reactance of Z_a.

Since a linear network can be reduced by Thévenin's theorem and the principle of superposition to a voltage source and a single series impedance, as in Fig. 9-3, the maximum-power-transfer theorem is quite general. Its application includes networks containing transmission lines.

Consider a simple transmission system as in Fig. 9-4. If we specify that the characteristic impedance of the line is pure resistance Z_0, and the load impedance Z_r is equal to Z_0, and the impedance of the sending-end apparatus Z_s is also Z_0, the condition for maximum power transfer is

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[3] *Proof*: Let $Z_a = R_a + jX_a$ and $Z_b = R_b + jX_b$.

$$P_b = I^2R_b = \frac{E^2}{(Z_a + Z_b)^2} R_b = \frac{E^2R_b}{(R_a + R_b)^2 + (X_a + X_b)^2} \qquad (9\text{-}16)$$

Varying X_b only, it is evident that P_b is maximum if $X_b = -X_a$. Adjusting X_b to this value, and holding it constant thereafter,

$$P_b = \frac{E^2R_b}{(R_a + R_b)^2} \qquad (9\text{-}17)$$

Differentiating with respect to R_b and equating to zero, to find the resistance condition for maximum power, gives $R_b = R_a$. Combining the two requirements, $Z_b = \overline{Z_a}$.

satisfied. Looking to the right at the terminals *ss'*, we see the input impedance to a line terminated in its characteristic impedance, Z_0. Looking to the left at *ss'*, we see the impedance of the sending-end apparatus, which is Z_0. Hence maximum power transfer is provided at the terminals *ss'*.

Looking to the right at the terminals *rr'*, we see the load impedance which is equal to Z_0. Looking to the left, we are again looking into a line with its characteristic impedance at the other end, and we see Z_0. Maximum power transfer is therefore provided at *rr'*. The transmission line can be divided into a left-hand part and a right-hand part by assuming an arbitrary division at any desired point, and looking either way from such a division point the impedance seen is always Z_0. Thus the condition for maximum power transfer is satisfied at every point along the line.

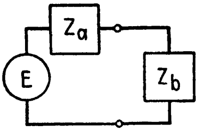

FIG. 9-3. Maximum power transfer.

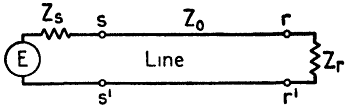

FIG. 9-4. Line with matching impedances.

Hence, if the characteristic impedance of a line is pure resistance, and if the line impedance is matched to the impedances of both sending-end and receiving-end apparatus, the highly desirable condition of maximum power transfer is obtained.[4]

In practice, approximations are usually permitted. The characteristic impedance of a telephone line is not, in general, a pure resistance. As illustrated in the curves of Fig. 5-2, the characteristic impedance of an open-wire line is a complex quantity with an angle of a few degrees. To avoid resonance on such a line it would be necessary to have a load with a small amount of capacitive reactance; to give maximum power transfer, it would be necessary to have a load with a small amount of inductive reactance. The practical solution is to use a load with resistance equal to the magnitude of the characteristic impedance of the line, neglecting angle.[5]

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[4] Impedance matching at all points is sufficient but not necessary for maximum power transfer. Consider that Fig. 9-4 represents a line without loss and $Z_s \neq Z_0$. Looking to the left at *rr'*, we do not see Z_0 but some other value Z. We then make $Z_r = \bar{Z}$; this gives maximum power transfer at *rr'*. Since there is no loss in the line, a condition that gives maximum power transfer at *rr'* must also give maximum power transfer at *ss'*. Impedance looking into the line at *ss'* is therefore matched to the generator impedance Z_s. Impedance looking into the line at *rr'* is Z_r; impedance looking into the line at *ss'* is Z_s; the line itself acts as an impedance-matching network as explained in Sec. 6. This arrangement, useful for radio antenna feeders, gives matching at one frequency only.

The characteristic impedance of a typical telephone cable circuit, however, is not even approximated by pure resistance. As seen in Fig. 5-5, the angle of the impedance is not far from 45 degrees. A satisfactory match to the impedance of a cable circuit (if not *loaded*) can only be provided by using a terminating network of resistive and reactive elements.

Figures 5-2 and 5-5 also show that the characteristic impedance of a line varies with frequency. A termination of pure resistance can have a magnitude equal to Z_0 at one frequency, but only one; and even a terminating network does not readily provide an exact match over the entire transmission range.

For a line transmitting at voice frequency, it is necessary to have an acceptable impedance match from about 250 to 3,000 cycles per second. A practical compromise is obtained by designing the line and load to match most precisely at 1,000 cycles. This frequency of 1,000 cycles is used for many design and test purposes as a typical telephone frequency. It is in the range to which the ear is most sensitive, the range most used in speech, and the range most necessary for clear articulation. If performance is good at 1,000 cycles, the telephone system will work reasonably well over the audio-frequency range.

For a carrier-current system, the problem is somewhat different. The characteristic impedance of open-wire line is almost constant over the frequency range used for carrier current, and even the characteristic impedance of cable, as seen in Fig. 5-5, is less variable than in the voice-frequency range. The lowest frequency band used for a telephone carrier system provides for two channels between about 4,000 and 10,000 cycles, and the variation of characteristic impedance of cable circuits through this range is not excessively great. For higher frequency channels, the variation is less.

We have seen that it is theoretically desirable to make the line impedance equal to the impedances of the load and generator. However, as a matter of fact, it is usually impractical to do anything of the kind. An amplifier at the receiving end of a long line, for instance, has an input impedance many times higher than the characteristic impedance of a line. Fortunately there are ways to overcome this difficulty and to provide a proper impedance match.

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[5] It can be shown by an extension of the proof in footnote 3 that if the magnitude of the load impedance can be varied, but the angle cannot, then maximum power will be received if the absolute value of the load impedance is equal to the absolute value of the impedance of the generating network.

6. Impedance Matching with Transformers. On telephone systems, impedance matching is done with transformers. It is well known that a transformer will change the apparent impedance, and that the ratio of voltage to current on one side of a transformer is related to the ratio of voltage to current on the other side by the square of the turn ratio of the transformer windings.

Consider the transformer shown in Fig. 9-5. Assuming this transformer is ideal, $V_2 = nV_1$ and $I_2 = I_1/n$. Impedance looking to the right at terminals 1-1 is V_1/I_1. Impedance looking to the right at terminals 2-2 is $V_2/I_2 = n^2V_1/I_1$.

If the transformer were omitted, and a direct connection were made between terminals 1-1 and terminals 2-2, the impedance looking to the right at terminals 1-1 would be n^2 times as great as it is with the transformer in use. This is the meaning of the statement that a transformer (often called a *repeating coil* in telephone parlance) transforms the impedance by the square of the turn ratio.

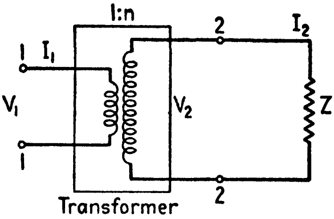

FIG. 9-5. Impedance matching by use of a transformer.

The use of a transformer in impedance matching is obvious. Suppose terminals 1-1 in Fig. 9-5 are to be connected to the end of a 600-ohm line (Z_0 = 600 ohms) and Z represents the input impedance of 9,600 ohms to an amplifier. (These values are reasonably typical of practice.) These impedances are in the ratio of 1 to 16. If a transformer is used as in the figure with a turn ratio of 1 to 4, the transformer coil with the larger number of turns being connected to the higher impedance, the line will be terminated in its characteristic impedance.

A transformer or *repeating coil* will thus serve to match a load to a line, provided the two impedances have the same phase angle. If, however, the phase angle of the load is not the same as that of the line, further adjustment is needed for a perfect match. By adding either inductance or capacitance at the load terminals, as may be needed, the load can be given the correct phase angle, and a transformer will then serve to match magnitudes. If a match is required at one frequency only, the reactance added for angle correction can be connected either in series or parallel with the load, and it may be on either side of the transformer. However, if a match is needed over a range of frequency, some ingenuity may be necessary to provide a terminating network with satisfactory impedance-frequency characteristics.

This discussion has assumed an ideal transformer. An actual transformer, having resistance, core loss, leakage reactance, magnetizing admittance, and distributed capacitance, will provide something less than a perfect match. Above moderate radio frequencies, core loss is so great that transformer matching can be used only with difficulty if at all. A matching network is then used. The great advantage of the simple transformer is that it is not sensitive to frequency and that it provides practically the same impedance transformation ratio over the entire range of frequencies for which it can be used.

Impedance matching on telephone systems is done with transformers, not only at voice frequency, but also for carrier-current operation as high as 140 kilocycles per second. It is not practical to design either transmitting or receiving equipment to match line impedance, and impedance-matching transformers are commonly used at both ends of lines. An impedance of 600 ohms is, in a sense, the standard impedance for telephone system operation. This is approximately the characteristic impedance of an open-wire telephone line at 1,000 cycles (with 165-mil wire, 12-inch spacing, $Z_0 = 612$ ohms; with 128-mil wire, 8-inch spacing, $Z_0 = 603$ ohms) although there are many lines with widely different impedance.

7. Impedance-matching Networks. It is possible to use a network of reactive elements instead of a transformer for impedance matching. Matching networks cannot readily be used for voice-frequency transmission, as they are satisfactory for only one frequency or, at most, a narrow band. They can be used for carrier-current service if the carrier frequency is high enough to make the voice band relatively narrow. They can be used for impedance matching at radio frequency. For example, the impedance of a radio broadcast antenna is frequently matched to the characteristic impedance of the transmission line leading to the antenna by means of a reactive network.

L-section Network. If, for example, a line of characteristic impedance Z_0 is to be terminated in apparatus with an input impedance R, as shown in Fig. 9-6, a network of reactances can be used between the line and the load with elements so chosen that the impedance across the receiving terminals of the line is (at one frequency) equal to Z_0.

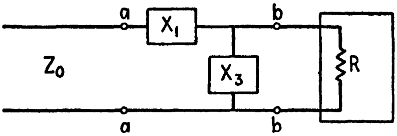

FIG. 9-6. Impedance matching by use of a reactive network.

As a simple numerical example, consider a line for which Z_0 is 400 ohms, required to supply power to apparatus with an input impedance of $R = 600$ ohms. Let us try to match impedances by inserting two reactances, X_1 and X_3, as shown in Fig. 9-6.

We know that impedance looking to the right at terminals *aa* is

$$Z_{aa} = jX_1 + \frac{R(jX_3)}{R + jX_3} = \frac{RX_3^2 + j[X_1X_3^2 + R^2(X_1 + X_3)]}{R^2 + X_3^2} \qquad (9\text{-}18)$$

Since this impedance looking in from the line should be equal to Z_0, its real component should be equal to 400 ohms, and its imaginary component should be zero. We therefore write

$$\frac{RX_3^2}{R^2 + X_3^2} = Z_0 = 400 \tag{9-19}$$

$$X_1X_3^2 + R^2(X_1 + X_3) = 0 \tag{9-20}$$

Equation 9-19 is readily solved for X_3:

$$X_3 = \pm\sqrt{\frac{Z_0}{R - Z_0}}\,R \tag{9-21}$$

and Eq. 9-20 gives

$$X_1 = \frac{Z_0 - R}{R}X_3 = \mp\sqrt{Z_0(R - Z_0)} \tag{9-22}$$

Introducing the numerical values for Z_0 and R,

$$X_3 = \pm 848 \text{ ohms} \qquad X_1 = \mp 283 \text{ ohms} \tag{9-23}$$

By using the values so obtained for X_1 and X_3, the impedance looking into the network at terminals *aa* is 400 ohms and the line is terminated in its characteristic impedance. Note that X_3 may be either a condenser or an inductance with reactance (at the frequency for which the impedances are matched) of 848 ohms; X_1 must be the inverse reactance, inductive or capacitive respectively, with a value of 283 ohms.

It is important that this network, which transforms a 600-ohm load on terminals *bb* to a 400-ohm impedance at terminals *aa*, gives the inverse transformation when viewed in the opposite direction. That is, if a 400-ohm load is attached to the terminals *aa*, the impedance looking in at *bb* is 600 ohms. A numerical solution for impedance looking in at *bb* gives

$$Z_{bb} = \frac{(400 + jX_1)jX_3}{400 + j(X_1 + X_3)} = \frac{(400 \mp j283)(\pm j848)}{400 + j(\mp 283 \pm 848)} = 600 \text{ ohms} \tag{9-24}$$

This is an illustration of a general principle. To derive the principle, first consider a theorem that applies to any network or series of networks of purely reactive elements: if an impedance match to give maximum power transfer (a conjugate impedance match) is provided at terminals *aa*, there will also result a perfect (conjugate) match at terminals *bb*. This follows from a line of reasoning used before:[6] if there is a perfect match at *aa* in Fig. 9-6, maximum power passes those terminals, and no adjustment of impedances to the right of *aa* can increase the power. Since the matching network is purely reactive, power at *bb* is the same as power at *aa*. If a change of impedance at *R* cannot increase power passing *aa*, it cannot increase power passing *bb*. Hence an impedance match at *aa* provides maximum power at *aa*, and also at *bb*, which must necessarily correspond to a conjugate impedance match at *bb*.

---

[6] See footnote 4.

This theorem tells us that if a network of reactive elements, with two pairs of terminals, is matched to the impedance connected at one pair of terminals, it is matched to the impedance at the other pair also. Hence, if a network of pure reactance elements transforms an impedance from Z_m to Z_n when power flows in one direction, it will transform from Z_n to Z_m when power flows in the other direction.

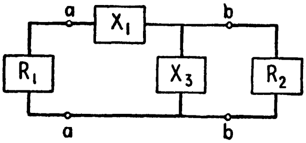

FIG. 9-7. An L matching network.

Since the network may be used in either direction, it is convenient to generalize as in Fig. 9-7, not specifying which direction power is to flow. The design equations for the network, to provide a match to both of the resistances R_1 and R_2 (taken from Eqs. 9-21 and 9-22 with changes of symbols only), are

$$X_1 = \mp\sqrt{R_1(R_2 - R_1)}$$

$$X_3 = \pm R_2\sqrt{\frac{R_1}{R_2 - R_1}} \qquad (9\text{-}25)$$

Note that R_2 in both equations is necessarily greater than R_1 (for a matching network to be made of reactive elements); this means that the matching network must be faced one way or the other as is necessary to put the shunt element X_3 next to the higher resistance.

These design equations assume that R_1 and R_2 are purely resistive. If the terminating impedances in Fig. 9-7 have reactive components, an inverse reactive element (having equal but opposite reactance at the frequency for which the network is designed) should be used. Thus, if the right-hand load were inductive, a condenser could be connected across the terminals *bb*, and R_2 is then the purely real combination of inductive load and condenser in parallel. This additional condenser does not need to be a distinct element, and can be combined with X_3 in the actual construction of the network. In a similar manner, if the left-hand impedance were inductive, the necessary correction could be made by connecting a condenser in series; the series condenser can be combined with X_1 in the actual physical network.

X_1 in Eqs. 9-25 can be either positive or negative, either inductive or capacitive, as the designer prefers. X_3 must then be the inverse reactance, with opposite sign. The designer's choice may be based on convenience or economy, but there is another electrical factor in the problem, not yet discussed, that may be important. This is the phase of the voltage delivered by the matching network, relative to the phase of the input voltage. (Current has the same phase shift between input and output as has the voltage, for both input and output impedances are purely resistive.)

It is not difficult to determine the change of phase as a signal passes through the network. Consider a signal passing from right to left in Fig. 9-7, and write the voltage at *aa* in terms of the voltage at *bb*:

$$V_{aa} = V_{bb}\frac{R_1}{R_1 + jX_1} = \frac{V_{bb}R_1}{R_1^2 + X_1^2}(R_1 - jX_1) \tag{9-26}$$

It is evident that V_{aa}, the output voltage, will lag V_{bb}, the input voltage, by an angle that may be called β such that

$$\tan\beta = \frac{X_1}{R_1} \tag{9-27}$$

For design purposes it is more convenient to substitute the first of Eqs. 9-25 for X_1 and derive

$$\cos\beta = \sqrt{\frac{R_1}{R_2}} \tag{9-28}$$

If X_1, the series element, is inductive, β is positive and phase is retarded (the output voltage lags).

If X_1, the series element, is capacitive, β is negative and phase is advanced (the output voltage leads).

Now consider a signal passing through the same network from left to right. A similar but slightly more involved solution for β gives, surprisingly, the same result. That is, Eq. 9-28 and the statements beneath it still apply; the phase shift is the same in amount and in sense for a signal passing in either direction.

A qualitative picture of why the L section is able to produce an impedance transformation is as follows. Consider the network of Fig. 9-6, by which a 600-ohm resistance R is to be transformed to a 400-ohm resistance at the terminals *aa*. The resistance R and the reactance X_3 are in parallel. The designer may choose whether X_3 shall be inductive or capacitive; let us decide that it is to be a condenser. The parallel combination of X_3 and R are then as shown in Fig. 9-8*a*.

This parallel combination of capacitance and resistance is equivalent (at one frequency) to a series circuit as shown at *b*, with capacitance C' and resistance R'. C' and R' must be such that the impedance of the series combination is equal to the impedance of the parallel combination; the value of R' depends on X_3 as well as on R, and R' is always less than R. Let us choose a value for X_3 that will make R' equal 400 ohms.

Now, C' and R' in series constitute a capacitive impedance, the real component of which is 400 ohms. By putting series inductive reactance into the circuit (as X_1 in Fig. 9-8*d*) equal and opposite to the reactance of C', we leave zero reactance, and obtain 400 ohms input resistance at the terminals *aa*. Figure 9-8*d* is a hypothetical network that is equivalent to the actual arrangement of elements as shown at *c*. This concept of the action of a matching network will be useful later, in connection with microwave impedance matching.

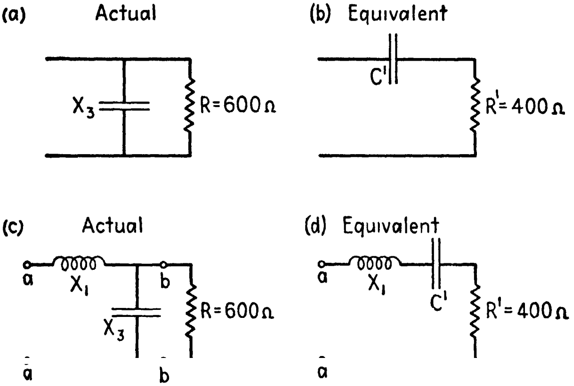

FIG. 9-8. Principle of operation of a matching network.

T- and Π-section Networks. The L-section matching network consists of two reactances, and there are only two variables in its design. In the above discussion, these design variables have been adjusted to match resistance to resistance and reactance to reactance. Since there are only two variables for the designer to work with, nothing more can be done—

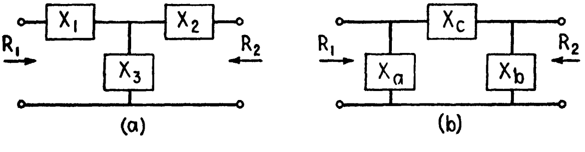

FIG. 9-9. T and Π matching networks.

except that the designer has the additional choice of making the series element inductive to provide phase retardation, or capacitive, to provide phase advance. The amount of the phase change is not under his control.

For some purposes it is necessary to provide current at the end of a transmission line with a specific phase relative to other currents or voltages. An example is seen in a transmission line used to feed a radio antenna if the antenna is part of an array. The designer is able to control phase as well as to provide proper matching if he uses a network comprising three reactive elements. These can be arranged as a T section (Fig. 9-9*a*) or as a Π section (Fig. 9-9*b*).

Design equations to match the T and Π networks to terminal resistances R_1 and R_2 and to introduce a phase lag β are as follows:[7]

For T section:

$$X_1 = -\frac{R_1 \cos\beta - \sqrt{R_1R_2}}{\sin\beta}$$

$$X_2 = -\frac{R_2 \cos\beta - \sqrt{R_1R_2}}{\sin\beta}$$

$$X_3 = -\frac{\sqrt{R_1R_2}}{\sin\beta} \tag{9-29}$$

For Π section:

$$X_a = \frac{R_1R_2 \sin\beta}{R_2 \cos\beta - \sqrt{R_1R_2}}$$

$$X_b = \frac{R_1R_2 \sin\beta}{R_1 \cos\beta - \sqrt{R_1R_2}}$$

$$X_c = \sqrt{R_1R_2}\sin\beta \tag{9-30}$$

The L-section network is a special case of either the T or the Π. Letting $\cos\beta = \sqrt{R_1/R_2}$ (Eq. 9-28), X_2 for the T section becomes zero and X_1 and X_3 reduce to the simpler values given in Eqs. 9-25 for the L section. (The choice of + or − sign in the latter equations arises from the choice of + or $-\beta$ in Eqs. 9-29, the sign of $\sin\beta$ not being determined by the sign of $\cos\beta$.) Equations for the Π section reduce similarly.

If the greater flexibility of a three-element network is not required to provide a predetermined phase shift, it can be used for other purposes. A T or Π network may be designed, for instance, to come closer to providing an impedance match over a range of frequencies than is possible with the two-element L-section network.

8. The Distortionless Line. It is to be expected that a line, even though properly terminated, will attenuate and distort the signal to some extent. Attenuation results from loss of energy in the line, and cannot be avoided, but amplifiers at the line terminals, and at intermediate repeater stations,[8] will compensate for attenuation. However, a transmission line does not ordinarily attenuate all frequencies equally, and unequal attenuation results in *frequency distortion.*

---

[7] From F. E. Terman, "Radio Engineering," p. 102 (Ref. 33).

[8] A telephone "repeater" is an amplifier with associated circuits; it is not to be confused with a telephone "repeating coil," which is a transformer in the language of telephone engineering.

It is theoretically possible to design a line that avoids distortion by attenuating all frequencies in the same proportion. Consider a line with the constants so related that

$$\frac{r}{g} = \frac{l}{c} \tag{9-31}$$

or

$$rc = lg \tag{9-32}$$

Such a line is called a *distortionless line,* and it has a number of interesting and desirable characteristics, including the following.

Characteristic impedance is purely real. By definition (Eq. 1-38),

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \tag{9-33}$$

Substitution from Eq. 9-32 gives

$$Z_0 = \sqrt{\frac{lg/c + j\omega l}{g + j\omega c}} = \sqrt{\frac{(g + j\omega c)l/c}{g + j\omega c}} = \sqrt{\frac{l}{c}} \tag{9-34}$$

It is evident that this is a real quantity.

Characteristic impedance is independent of frequency. This also is evident from Eq. 9-34, for ω has disappeared from the equation. It follows from these two conclusions that the characteristic impedance of a distortionless line can be matched by a single pure resistance at all frequencies. A simple resistance termination gives only an approximate match for any nondistortionless line.

Attenuation is independent of frequency. Attenuation is given by Eq. 5-6:

$$\alpha = \sqrt{\tfrac{1}{2}[\sqrt{(r^2 + \omega^2 l^2)(g^2 + \omega^2 c^2)} + rg - \omega^2 lc]} \tag{9-35}$$

From Eq. 9-32, $g^2 = r^2c^2/l^2$, which gives

$$\alpha = \sqrt{\frac{1}{2}\left[\sqrt{(r^2 + \omega^2 l^2)(r^2 + \omega^2 l^2)\frac{c^2}{l^2}} + rg - \omega^2 lc\right]}$$

$$= \sqrt{\frac{1}{2}\left[\left(r^2\frac{g}{r} + \omega^2 l^2 \frac{c}{l}\right) + rg - \omega^2 lc\right]} = \sqrt{rg} \tag{9-36}$$

Frequency, as a factor of ω, has disappeared from this equation, leaving α independent of frequency.

Attenuation is minimum. It is obvious that loss on a transmission line can be reduced by decreasing the resistance r or the leakage g. If it is assumed that r and g cannot be reduced, there is a best relation of inductance l and capacitance c to give minimum attenuation. This turns out to be the same relation as that for the distortionless line. The first step in the demonstration is to differentiate α in Eq. 9-35 with respect to l and set the result equal to zero; this is a moderately involved differentiation, but not at all difficult. Then let

$$\frac{d\alpha}{dl} = 0 \tag{9-37}$$

which gives

$$l(g^2 + \omega^2c^2) - c\sqrt{(r^2 + \omega^2l^2)(g^2 + \omega^2c^2)} = 0 \tag{9-38}$$

from which

$$l^2(g^2 + \omega^2c^2) = (r^2c^2 + \omega^2l^2c^2) \tag{9-39}$$

Dividing both sides by l^2,

$$g^2 = \frac{r^2c^2}{l^2} \tag{9-40}$$

Hence this relation, which is also the relation for the distortionless line, gives minimum attenuation as a function of l.

An exactly similar differentiation shows that the same relation gives minimum attenuation as a function of c. The distortionless line therefore has the least attenuation that can be attained by variation of inductance or capacitance or both. The amount of the attenuation is given by Eq. 9-36.

Velocity of propagation is independent of frequency. It was shown in Eq. 1-49 that the velocity of propagation on a transmission line is ω/β. On the distortionless line, $\beta = \omega\sqrt{lc}$; that is, *the phase constant is proportional to frequency.* This is shown by treating Eq. 5-7 for β as Eq. 5-6 for α is handled in obtaining Eq. 9-36, whereupon Eq. 5-7 reduces to

$$\beta = \omega\sqrt{lc} \tag{9-41}$$

We therefore have

$$\text{Velocity} = \frac{\omega}{\beta} = \frac{1}{\sqrt{lc}} \tag{9-42}$$

Velocity of propagation is maximum. Any combination of constants different from the relation for the distortionless line gives a velocity of wave propagation less than that specified by Eq. 9-42. This can be shown by differentiating β in Eq. 5-7 with respect to any one of the constants r, g, l, or c (as α was differentiated in obtaining Eq. 9-38) and setting equal to zero. The relation $rc = lg$ results. Hence β is minimum and velocity is maximum on the distortionless line.

Because attenuation and velocity of propagation are both independent of frequency, a line characterized by the relation $rc = lg$ does not introduce either *frequency distortion* or *delay distortion*. On a properly terminated line, the received signal will be identical in form to the transmitted signal. The magnitude of the signal will be reduced by attenuation, but every part of the signal will be reduced in proportion. (It is shown in Chap. 14 that the same statement can be made of transient traveling waves.) This is the reason for the name of the *distortionless line*.

The six characteristics of the distortionless line listed above are all desirable, and a distortionless line would be particularly good for telephone service. Yet practical lines are never distortionless. The insulation conductance g is so small on actual lines that the necessary ratio of other parameters cannot practically be attained.

In considering how a line could be made distortionless, the first obvious suggestion is to increase g. To do so would increase loss, however, and attenuation would be greater. This disadvantage far outweighs any possible advantages.

The next suggestion, noting that r/g should equal l/c, would be to make r small. This requires larger wires, and the expense is prohibitive.

Larger inductance or smaller capacitance would tend toward the distortionless relation. Inductance can be increased, and capacitance decreased, by using greater spacing between wires. Greater spacing requires more expensive pole and crossarm structures, and this also is prohibitive. (Indeed, for open-wire carrier circuits, rather close spacing of the two wires of a pair is desirable, together with maximum practical spacing between circuits, to reduce *crosstalk*.)

Another possible way to increase inductance is to increase the magnetic permeability of the magnetic field. If some magnetic material were placed in the space close around the conductors of a telephone line, inductance would be increased. Permalloy tape has been wrapped around the wires of submarine telephone cable to give what is called *continuous loading*, to increase inductance, but the expense is too great for ordinary telephone installations. A more practical solution is to use *loading coils*; these are small inductance coils placed in each wire of a telephone line at frequent intervals.

9. Loaded Cable Circuits. Loading coils are used in telephone cable circuits. They provide inductance with a minimum of resistance and core loss. They are wound on permalloy dust cores (usually molybdenum-permalloy) and have inductance of the order of magnitude of 10 to 100 millihenrys. The unavoidable resistance is a few ohms.[9] It is evident that they introduce lumps of inductance into the line at finite intervals, instead of increasing the distributed inductance of the line in a uniform manner, but for some purposes this kind of "lumpy" loading is entirely satisfactory.

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[9] Typical values are 22 millihenrys and 1.25 ohms for coils to be spaced 3,000 feet (Type B-22); or 88 millihenrys and 7.3 ohms for coils to be spaced 6,000 feet (Type H-88) (see Ref. 34).

Loading coils provide a means of introducing a fairly large amount of inductance into a line. The practical effect is much the same as if the distributed inductance were increased, provided the coils are placed in the line at distances not much greater than a quarter-wavelength of the highest frequency transmitted.

Loading coils are advantageous for the transmission of the low frequencies. However, loading coils, acting with the capacitance and inductance of the line, constitute a low-pass filter, and although a loaded line will very satisfactorily transmit frequencies below what is called the cutoff frequency, all higher frequencies are severely attenuated. Thus, a line on which heavy lumped loading is employed will not transmit the frequencies required for carrier-current transmission.

From the practical point of view, nearly all the parameters of a circuit are improved by loading, for frequencies below cutoff. Impedance per unit length, z, is increased, and consequently the characteristic impedance Z_0 is raised. (Typical values for a cable circuit are given in Table 9-2.)

TABLE 9-2
CHARACTERISTICS OF TYPICAL TELEPHONE CIRCUITS
AT 1,000 CYCLES PER SECOND*

	Per loop-mile				α per mile	β per mile	Z_0, ohms	Velocity, miles per second	Decibels per mile
	r, ohms	l, henrys	c, μf	g, μ mhos					
Nonloaded cable†	85.8	0.001	0.062	1.5	0.125	0.134	$470\underline{/-42.80^\circ}$	46,900	1.08
Loaded cable‡	92.2	0.078	0.062	1.5	0.0418	0.439	$1{,}131\underline{/-5.22^\circ}$	14,320	0.36
Open wire§	10.15	0.00366	0.00837	0.29	0.00760	0.0355	$692\underline{/-11.75^\circ}$	177,000	0.066

* Data courtesy of American Telephone and Telegraph Company.

† No. 19 AWG nonloaded side circuit, paper insulated.

‡ Same cable with H-88-S type loading: Coil spacing, 6,000 feet; coil inductance, 88 millihenrys; coil resistance, 7.3 ohms (cutoff frequency, approximately 4,000 cycles).

§ 104-mil wire, 12-inch spacing, non-pole-pair side circuit of a line carrying 40 wires, dry weather, temperature 20°C.

Z_0 becomes more nearly resistive, and more nearly uniform over the range of frequencies transmitted. This helps with impedance matching. The higher z results in a higher phase constant β, and therefore in lower velocity of propagation. Lower velocity is a disadvantage; but loading tends to equalize the velocity of propagation at different frequencies, which tends to avoid delay distortion, and this is an advantage.

Table 9-2 gives parameters of typical practical circuits at a frequency of 1,000 cycles. Figure 9-10a shows the marked reduction of attenuation of transmitted frequencies that results from loading, the increase of attenuation as the cutoff frequency is approached and exceeded, and Fig. 9-10b shows the uniform velocity of propagation of transmitted frequencies in a loaded cable.

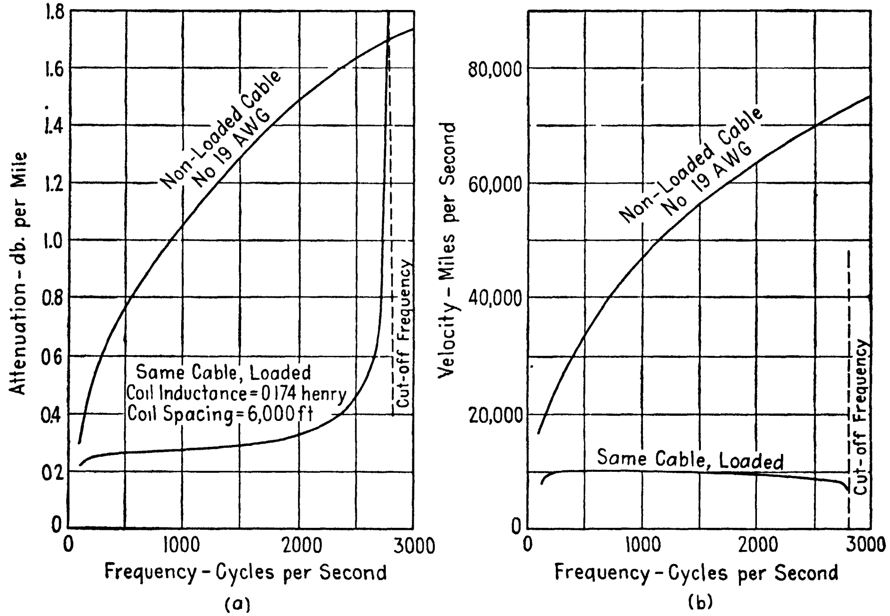

FIG. 9-10. Attenuation and velocity in a heavily loaded cable circuit, showing disadvantage of too heavy loading.
Nonloaded No. 19 AWG cable:

r = 85 ohms per loop-mile
l = 1.0 millihenry per mile
c = 0.062 microfarad per mile
g = 1.5 micromhos per mile

It has been said that the effect of loading coils is much the same as if an equal amount of inductance were introduced into the line with uniform distribution, provided the frequency is well below cutoff. For many computations, it is satisfactory to assume that the added impedance is actually distributed; the total equivalent inductance per foot of line is then found by dividing the inductance of the coil by the number of feet between coils and adding this to the actual inductance per foot of line. Parameters given in Table 9-2 are computed on this basis.

To show that attenuation is reduced by increasing the inductance of a cable, consider Eq. 5-16. If leakage is small (as it is in both open-wire and cable lines) so that it is a good approximation to let $g = 0$, Eq. 5-16 becomes

$$\alpha \approx \frac{rc}{2\sqrt{lc}} \tag{9-43}$$

This is more commonly written

$$\alpha \approx \frac{r}{2\sqrt{l/c}} \tag{9-44}$$

This expression shows that if lg is much smaller than rc, an increase of inductance will reduce the attenuation. (It may be conceived as increasing the voltage at which energy is transmitted.) Decrease of attenuation resulting from increased inductance is the important practical reason for loading cable circuits.

10. Computation of Parameters of Loaded Lines. The assumption that a loaded line acts as if the inductance of the coils were actually distributed is oversimplified for some purposes. It will not permit computing attenuation near the cutoff frequency. It does not indicate the frequency of cutoff. Indeed, it does not even suggest that there is a cutoff frequency.

Problems relating to cutoff are attacked as follows. Figure 9-11*a* shows two of the loading coils that are inserted with equal spacing along a line. (Loading coils are actually put in both wires of the line to maintain balance to other circuits and to ground, but this makes no difference to the analysis. For computation, we assume the total inductance and resistance of the coils to be in series with the line; whether in one wire or in two is immaterial.) Let us now associate with the length of line shown in the figure one-half of the inductance and resistance of each of the coils shown. Thus

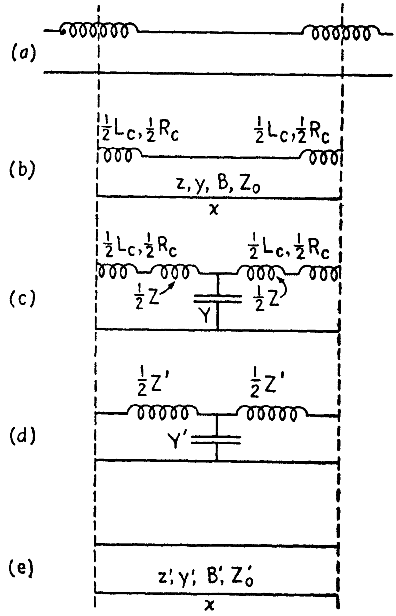
(a)
(b)
$\frac{1}{2}L_c, \frac{1}{2}R_c$
$\frac{1}{2}L_c, \frac{1}{2}R_c$
z, y, B, Z_0
x
(c)
$\frac{1}{2}L_c, \frac{1}{2}R_c$
$\frac{1}{2}L_c, \frac{1}{2}R_c$
$\frac{1}{2}Z$
Y
$\frac{1}{2}Z$
(d)
$\frac{1}{2}Z'$
$\frac{1}{2}Z'$
Y'
(e)
z', y', B', Z_0'
x

FIG. 9-11. Steps in computing the action of a loaded line.

each section of loaded line consists of the actual line plus half a loading coil at each end. This is indicated in Fig. 9-11*b*, where L_c and R_c are the inductance and resistance of a loading coil, z and y are the impedance and admittance per unit length of the line, and B and Z_0 have their usual meanings:

$$B = -j\sqrt{zy} \qquad Z_0 = \sqrt{\frac{z}{y}} \tag{9-45}$$

The length of line between loading coils is x.

We wish to know how this composite section of line and loading coil behaves. The solution proceeds in three steps. First, the actual line of Fig. 9-11*b* is changed to an equivalent artificial line, as in *c* of the figure. Second, inductance and resistance of the actual loading coil are combined with inductance and resistance of the artificial line, giving, as in *d*, an artificial line section that is equivalent to line-plus-loading-coil. Third, this derived artificial line section is converted back into a line with distributed constants, as in *e*, having parameters B' and Z_0', and from these parameters we can deduce the information that we need.

Formulas for transformation from a smooth line to an artificial line (*b* to *c*) were developed in Chap. 8 (see Table 8-1), as were also formulas for the inverse transformation from an artificial line to a smooth line (*d* to *e*). They are applied as follows, using the symbols of Fig. 9-11. In Fig. 9-11*c*,

$$\frac{1}{2}Z = \frac{zx}{2}\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} \tag{9-46}$$

$$Y = yx\frac{\sin Bx}{Bx} \tag{9-47}$$

In Fig. 9-11*d*,

$$\tfrac{1}{2}Z' = \tfrac{1}{2}Z + \tfrac{1}{2}Z_c = \tfrac{1}{2}Z + [\tfrac{1}{2}R_c + j\omega(\tfrac{1}{2}L_c)] \tag{9-48}$$

$$Y' = Y \tag{9-49}$$

In Fig. 9-11*e*,

$$\cos B'x = 1 + \frac{1}{2}Z'Y' = \cos Bx + j\frac{Z_c}{2Z_0}\sin Bx \tag{9-50}$$

$$Z_0' = \sqrt{\frac{Z'}{Y'} + \frac{Z'^2}{4}} \tag{9-51}$$

From B' in Eq. 9-50, α' and β' are found, for

$$B' = \beta' - j\alpha' \tag{9-52}$$

Attenuation and velocity of the signal in the loaded line are thereby determined.

Let us compute cutoff frequency. The cutoff frequency of a line is not much affected by line losses, so we shall neglect losses in both line and loading coils. This simplifies the work, without decreasing its practical value. Before going further, it will be well to consider exactly what is meant by *cutoff frequency.*

If line and loading coils have no loss, attenuation below cutoff frequency is zero. Above cutoff frequency, there is attenuation. This is the definition of cutoff frequency.

In our work with smooth lines, we have become accustomed to thinking of attenuation as a result of power loss. The idea of attenuation on a lossless line is therefore somewhat surprising. For the present, it will be sufficient to consider that attenuation as used in the present discussion means not power loss, but power rejection. If a section of loaded line, even though lossless, refuses to pass energy, so that a signal applied to such a section is not delivered by it, the signal is blocked as effectually as if the energy were wasted in resistance. (The resulting attenuation, sometimes called *reactive attenuation*, is more fully discussed in Chap. 10.)

The following mental picture[10] is helpful in visualizing the blocking effect of a loaded line. A mechanical traveling wave may be carried by a stretched cord, as in Fig. 9-12*a*. To enable the vibrating cord to carry more energy with the same amplitude of vibration, it can be loaded by attaching to it a large number of buckshot as in Fig. 9-12*b*. The effect is quite different if a few large lumps of lead are attached, as in Fig. 9-12*c*. Figure 9-12*c* illustrates the blocking action of coils spaced too far apart to pass the frequency applied; Fig. 9-12*b* shows the transmission of a frequency in the pass band. Although in Fig. 9-12*c* energy is not delivered

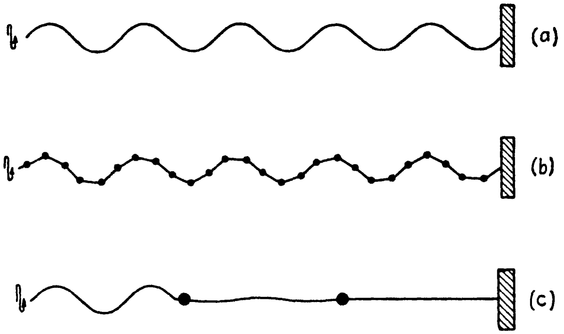

FIG. 9-12. "Loading" a mechanically vibrating cord with small weights.

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[10] Adapted from "Principles of Electricity Applied to Telephone and Telegraph Work," American Telephone and Telegraph Company (Ref. 34).

to the right-hand end of the line, neither is it dissipated in the line; it is merely rejected because the heavy lump of lead refuses to move in response to the vibration of the light cord, and the wave is reflected back to the source. For the wave to be transmitted, the lumps must be close enough together to pull each other up and down, not relying on the mass of the cord.

Returning to the computation of cutoff frequency, when losses are neglected, Eqs. 9-50 and 9-51 become

$$\cos B'x = 1 + \frac{1}{2}(j\omega L')(j\omega C') = 1 - \frac{\omega^2 L'C'}{2} \tag{9-53}$$

$$Z_0' = \sqrt{\frac{L'}{C'} - \frac{\omega^2 L'^2}{4}} \tag{9-54}$$

Equation 9-53 gives the cutoff frequency, by the following argument. If $B'x$ is real, β exists, α does not, and a signal is transmitted without attenuation (see Eq. 9-52). If, on the other hand, $B'x$ is imaginary, α exists, β does not, and there is attenuation. Now Eq. 9-53 can have a real solution for $B'x$ only if the right-hand member of the equation has a value between 1 and -1, for the cosine of a real variable has values only in this range. Hence, if the value of $1 - \frac{1}{2}\omega^2 L'C'$ is between 1 and -1, $B'x$ is real and there is no attenuation; while if the value is greater than 1 or less than -1, $B'x$ is not real and there is attenuation rather than transmission.

Since L' and C' are positive, the quantity $1 - \frac{1}{2}\omega^2 L'C'$ cannot be greater than 1, but it can be less than -1. The frequency at which it becomes -1 is the cutoff frequency, for at lower frequencies there is no attenuation, whereas at higher frequencies there is attenuation. Consequently, the cutoff frequency is that for which

$$1 - \tfrac{1}{2}\omega^2 L'C' = -1 \tag{9-55}$$

If the cutoff frequency is called f_c, and the corresponding ω is ω_c, we have

$$\omega_c = \frac{2}{\sqrt{L'C'}} \tag{9-56}$$

or

$$f_c = \frac{1}{\pi\sqrt{L'C'}} \tag{9-57}$$

The sudden increase of attenuation in an actual circuit as the cutoff frequency is exceeded is shown in Fig. 9-10a.

Since the fractions $\tan \frac{1}{2}Bx/\frac{1}{2}Bx$ and $\sin Bx/Bx$ in Eqs. 9-46 and 9-47 are always very nearly equal to 1, cutoff frequency can be computed in practical problems by using the following approximate expressions in Eq. 9-57:

$$L' = lx + L_c \tag{9-58}$$

$$C' = cx \tag{9-59}$$

where l is distributed inductance (henrys) per unit length of the actual line
c is distributed capacitance (farads) per unit length of the actual line
x is length of line between loading coils
L_c is inductance (henrys) per loading coil

Note that Z_0' is imaginary at frequencies above the cutoff frequency. A line with an imaginary characteristic impedance cannot be expected to transmit power.

11. Equalizing Networks. Since it is not actually feasible to build a distortionless line, or even to load a practical line enough to minimize distortion, telephone engineers have sought an entirely different means of providing a distortion-free signal at the receiving end of a line. The general principle is to accept the unequal attenuation and phase shift that result on a nonloaded line or cable, and then to provide compensating attenuation and phase shift in terminating networks.

If a telephone line or cable attenuates the high frequencies more than the low frequencies, as is usual, the equalizing network employed with the line will attenuate low frequencies more than high frequencies, making the total attenuation of all frequencies substantially the same.

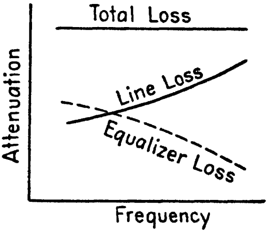


FIG. 9-13. Equalization of attenuation.

Ideal equalization is shown in Fig. 9-13. The curve shown for line loss is typical of an unloaded cable circuit. The range of frequency for which equalization is required may be only the audio-frequency band for voice transmission, but, if the cable is to be used for carrier transmission, it is necessary to equalize attenuation of a group of carrier channels. The width of the frequency band requiring equalization in a carrier-current cable circuit may be from 10 to as much as 50 kilocycles.

The engineering problem is to design an equalizing network with the desired attenuation. As indicated in Fig. 9-13, loss in the equalizer should be inverse, as a function of frequency, to the line loss; also, the minimum equalizer loss should be as low as possible to give minimum total loss. It is further necessary to provide an equalizing network that will not disturb the impedance match of the line.

There are several simple ways to obtain an approximate kind of equalization if the amount of attenuation required is not great. A condenser in series with the load, for example, reduces power to the load at low frequency more than at high, and hence provides an attenuation crudely similar to that indicated in Fig. 9-13. Another crude means of equalization is to shunt across the load an impedance element with resistance and inductance in series; this shunt will divert more energy from the load at low frequency. Either of these methods tends to give the desired equalization, but at the expense of spoiling the impedance match between load and line. But if both of these arrangements are used together, in the proper proportion, it is possible to get the desired attenuation while maintaining an impedance match. This leads to what is called the *bridged*-T *network*. (Equivalent T, Π, or lattice networks can be used instead of a bridged T, but they require more reactive elements in the networks, and hence are less desirable.)

Figure 9-14*a* shows a bridged-T network between a load of resistance R and a generator with internal resistance R. It is desired that the bridged

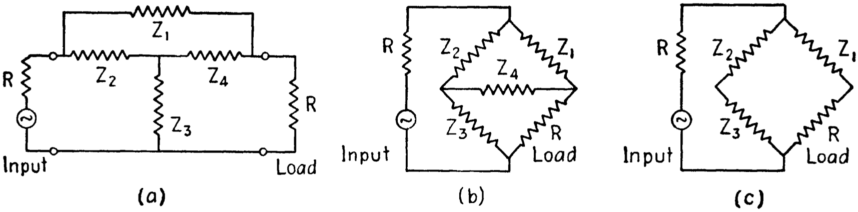


FIG. 9-14. Bridged-T equalizing network.

T be matched to the terminating impedance R at both ends. The bridged T itself must be symmetrical; that is, $Z_4 = Z_2$. Now, the same network can be drawn to have the appearance of a Wheatstone-bridge circuit as in Fig. 9-14*b*; this is electrically the same, but is arranged in a form that makes it clear that if

$$\frac{Z_2}{Z_3} = \frac{Z_1}{R} \tag{9-60}$$

there will be no current through Z_4. Let us specify that the bridged T is designed to have these proportions; Z_4 then carries no current and may be completely omitted from the diagram as in Fig. 9-14*c*.

It is evident from Fig. 9-14*c* that the input impedance to the bridged-T network when terminated with the load R is

$$Z_{in} = \frac{(Z_1 + R)(Z_2 + Z_3)}{Z_1 + Z_2 + Z_3 + R} \tag{9-61}$$

For a proper impedance match, this input impedance must equal R. We therefore set Eq. 9-61 equal to R and solve for Z_2. Combined with Eq. 9-60, this gives

$$Z_2 = R \tag{9-62}$$

Since, for symmetry, Z_4 must equal Z_2, it follows that $Z_4 = R$ also. From Eq. 9-60,

$$Z_1Z_3 = Z_2R = R^2 \tag{9-63}$$

Hence we find that for a bridged-T network to match R at both ends, it is necessary that Z_2 and Z_4 equal R, and that Z_1 and Z_3 satisfy Eq. 9-63.

This latter condition is sometimes expressed by saying that Z_3 must be the inverse of Z_1. If the equation is written

$$\frac{Z_1}{Y_3} = R^2 \tag{9-64}$$

it is evident that Z_1 and Y_3, to be inverse, must have the same proportion of real and imaginary components. Hence, if Z_1 is pure resistance, Y_3 must be pure conductance. If Z_1 is inductive, Y_3 must be capacitive. To tabulate the inverse relationship,

If one of the pair is:	*the inverse unit must be:*
Resistance	Resistance
Inductive reactance	Capacitive reactance
Capacitive reactance	Inductive reactance
Impedance units in series	The inverse units in parallel
Impedance units in parallel	The inverse units in series

Thus, if Z_1 is a condenser, Z_3 must be an inductance, and a bridged-T network so designed becomes that of Fig. 9-15. Remembering that the element corresponding to Z_4 carries no current when a signal is being transmitted from left to right, it will be seen that this bridged T is a means of putting a condenser in series with the load, and shunting a coil with resistance and inductance across the load, while at the same time maintaining an exact impedance match.

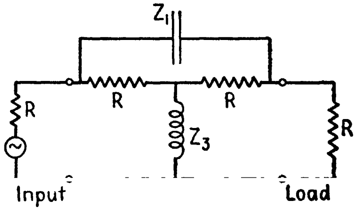


FIG. 9-15. A crude example of a bridged-T equalizing network.

Figure 9-15 is still a crude and inadequate network for equalizing attenuation. In general, Z_1 should not be a single element, but a carefully designed combination of many elements in parallel and series. Z_3 is the corresponding inverse network. It is apparent (remembering, still, that Z_4 carries no current) that the voltage of the signal applied to the bridged-T network is applied to Z_1 and the load in series. Therefore Z_1, and Z_1 alone, determines the part of the applied voltage that reaches the load and hence the attenuation of the signal in the bridged-T network. If Z_1 (like a condenser) has a higher impedance to low frequencies, low frequencies are more attenuated by the bridged T. If Z_1 gives series resonance at a certain frequency, that particular frequency is passed by the network to the load without attenuation, whereas if Z_1 has parallel resonance, the resonant frequency is blocked. The proper choice of elements to compose Z_1 in order to provide the desired attenuation at all frequencies is the art of equalizing-network design.

As an alternative to compensating for unequal line loss by providing additional loss in an equalizing network, it is possible to compensate by providing unequal gain in an associated amplifier circuit. Negative feedback is used in the amplifier, and the equalizing network is put in the feedback circuit. The basic principle of equalization is clearly the same whether compensation is accomplished by providing more loss or less gain.

Pads. Networks are often required in telephone systems to attenuate all frequencies equally. Such networks are called *pads*, and are made of resistances only. A pad must not disturb the impedance match of the system; thus it is necessary that the impedance looking into the pad from either direction equal the characteristic impedance of the lines (or other apparatus) to which the pad is connected. A pad may take the form of a Π or T network. To design a pad, reference may be made to Chap. 10. Use Eq. 10-7 or 10-8 for characteristic impedance, referring to Fig. 10-5 and using pure resistances for the impedances; use Eq. 10-23 for the desired attenuation, again letting Z_1 and Z_2 be pure resistance; then solve these two relations simultaneously.

In lines that are balanced to ground, pads must take the *balanced* Π or *balanced* T forms of Fig. 10-27. These are also called *square* pads or H pads, respectively.

Time-delay Equalizer. A time-delay or phase-equalizing network is used to eliminate delay distortion introduced by the transmission line. For transmission of sound, delay distortion is important only for high-quality program transmission over loaded cable circuits. Delay distortion is not noticeable, even in high-fidelity muscial programs, unless the higher frequencies are delayed as much as 5 or 10 milliseconds more than the lower frequencies. For picture transmission, on the contrary, no delay distortion can be tolerated beyond the duration of one picture element—approximately one-quarter cycle of the highest frequency transmitted. Picture transmission, of course, includes television.

For delay equalization, a network is added at the end of the line that introduces additional delay, to bring the total delay for all transmitted frequencies to (ideally) the same value. There is then no delay distortion. The practical delay equalizer uses a number of bridged-T sections in series.

12. Amplification. Plainly, the use of attenuating networks in a telephone system is practicable only if there are ready means of amplification to make up for the power loss. If amplification were not cheap and easy, it would be necessary to prevent distortion by avoiding attenuation rather than to introduce additional attenuation for compensation. However, modern vacuum-tube techniques provide any reasonable amount of amplification with good fidelity at comparatively small cost.

A sharp distinction must be made between the cost of providing greater amounts of amplification at an already existing terminal or repeater (amplifier) station, and the cost of building additional repeater stations. Additional stations are expensive, both in first cost and maintenance. The distance between repeater stations along a line should be as great as other considerations will allow.

The determining factors are roughly as follows. The signal strength on a telephone line cannot be allowed to become too low at any point, for the signal level must be kept well above the noise level. A satisfactory *signal-to-noise ratio* must be maintained where the signal is weakest, and this will naturally be where the line enters a repeater station. Since a certain amount of noise is unavoidable in any circuit, this sets the lower limit of telephone transmission energy level.

"Noise" in a telephone circuit results from various sources. There is a very small amount of unavoidable noise in every amplifier. The general inductive effect of other unrelated signals in nearby circuits is a form of noise called "babble." This is particularly likely to be noticeable in cable circuits. Open-wire telephone lines are exposed to atmospheric disturbances and to induction from parallel power lines. Inductive interference from power lines is extremely important. It is minimized by avoiding long stretches of parallel construction of telephone and power lines and by transposing the wires of both power and telephone lines.

Noise within a cable is reduced by careful balancing and transposition of circuits. Noise from external sources may be diminished by transposition, and also by shielding the cable circuits. For shielding, cables may be wrapped with metal tape, using copper or iron tapes, or both.

"Crosstalk" is a form of interference that results when one circuit is disturbed primarily by one neighboring circuit. It results from mutual inductance or capacitance between circuits. Every effort is made in line construction to eliminate such mutual inductance and capacitance, using transposition and careful balancing of all circuits to ground. Also, the wires of each pair are kept close together.

In order to maintain a satisfactory signal-to-noise ratio, the input to a repeater station must not be too low. The output must not be too high for satisfactory and economical operation of the equipment. Also, too high an energy level is conducive to crosstalk in some types of circuits. Repeater gain is limited by the permissible difference between input and output levels. The practical amount of gain is dependent on the type of circuit; this is discussed in Sec. 15.

13. Special Devices. Two special devices, the hybrid coil and the phantom circuit, should be mentioned because of their interest. They are important in the practice, although they relate only indirectly to the theory, of long lines.

Hybrid Coil. A transmission line will carry signals in either direction. An amplifier, however, operates in one direction only. Hence, when amplifiers (repeaters) are used in a telephone line it is necessary either to restrict the operation of each pair of wires to a single direction only, or else to carry signals in both directions on a pair of wires and to provide some means of separating the east-west signal from the west-east signal at each repeater station for amplification.

The first thought would perhaps be to provide two amplifiers at each repeater station, one to operate in each direction, without any auxiliary equipment. This would mean, however, that the east-west amplifier would receive the output of the west-east amplifier, and the west-east amplifier would receive the output of the east-west amplifier, and there would be a regenerative circuit that would "sing," putting the line out of commission. To prevent "singing," the hybrid coil acts to prevent one amplifier from hearing what the other amplifier says.

The general principle is shown in Fig. 9-16. Figure 9-16*a* shows an eastbound signal. It enters the diagram at the output terminals of the west-east amplifier, at the word *In*. At the hybrid coil it divides; half the energy goes to a balancing network, and is wasted; the other half goes to the eastbound line at *Out* and carries the message on to the next repeater station.

The hybrid coil is essentially a transformer with three windings on one core. It can be seen from the figure that if the same voltage is induced in each half of the secondary of the coil, and if the impedance of the balancing network is exactly equal to the input impedance to the line, the condition is that of a balanced bridge and there is no potential between the terminals *a* and *b*. Hence there is no input to the east-west amplifier, although a signal is passing through the station from west to east.

When a signal arrives from the east, however, the operation is as shown in Fig. 9-16*b*. Half of the incoming voltage is applied to the input terminals of the east-west amplifier. The other half appears across the output terminals of the west-east amplifier where, except for wasting half of the received energy, it is of no significance. The east-west amplifier operates, and sends the signal on toward the west. The signal, after being strengthened by the east-west amplifier, enters the westbound line through another hybrid coil which prevents the west-east amplifier from being actuated, and the signal travels on westward to the next repeater station. The equipment at each repeater station is as shown in Fig. 9-17.

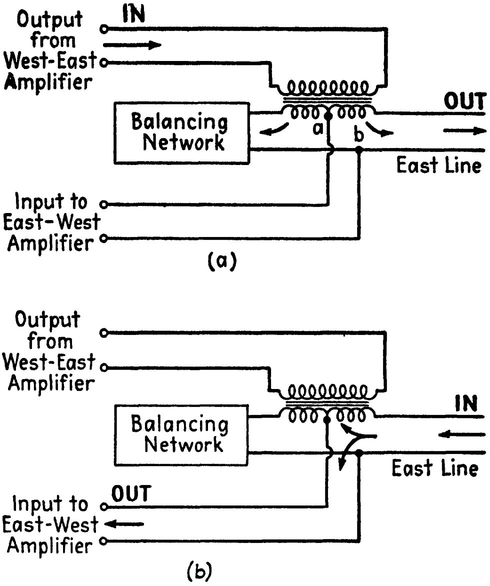

Output from West-East Amplifier
IN
Balancing Network
a
b
OUT
East Line
Input to East-West Amplifier
(a)
Output from West-East Amplifier
Balancing Network
IN
East Line
Input to East-West Amplifier
OUT
(b)

FIG. 9-16. The hybrid coil.

In the actual hybrid coil, each of the three windings is divided into two equal parts. The windings that go into the line circuit and into the balancing network circuit are divided, with half in each side of the line; this is to maintain balance to ground. The actual connection is therefore more nearly as shown by the standard schematic diagram of Fig. 9-17 than it is by the diagrams of Fig. 9-16, but the principle is the same.

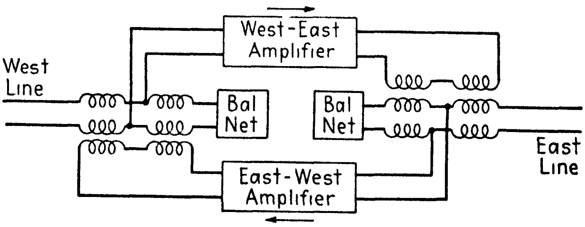


FIG. 9-17. Arrangement of hybrid coils in a repeater station.

It must be understood in considering the operation of the hybrid coil that all impedances are matched. That is, the output impedance of the west-east amplifier, the input impedance of the east-west amplifier, and the impedance of the balancing network are all matched to the impedance of the east line. Note, moreover, that if the east line were not properly terminated at its distant end, a reflection of an eastbound signal would be sent back from the mismatch, and this reflection would enter the east-west amplifier like any other westbound signal. It is because of the practical difficulty of obtaining perfect match and perfect balance that the amplification in a two-way repeater station must be limited to about 18 decibels gain.

Difficulties of matching and balancing are so great at high frequency that two-way transmission is frequently not used for carrier-current operation. In certain applications (Type K), the east-west signal is carried on one pair of wires, and the west-east signal on another pair in a separate cable. In other systems (Type J), the east-west signal is carried in one carrier-frequency band and the west-east signal in another, on the same pair of wires. In the latter system, separation of the two bands at a repeater is accomplished by band-pass filters. In either case, the east-west frequency band, containing 12 voice channels, is amplified at a repeater station by one amplifier, and the west-east band by another. Hybrid coils are used at terminal stations, where connections to the subscribers' circuits are made through hybrid coils that operate at voice frequency.

Phantom Circuit. A pair of wires provides one voice-frequency telephone channel. Two pairs of wires can be used to provide three channels. Each pair of wires provides an ordinary circuit, called a *side circuit*, and a so-called *phantom circuit* gives the third channel. The phantom circuit uses one pair of wires for one of its conductors and the other pair of wires for its other conductor.

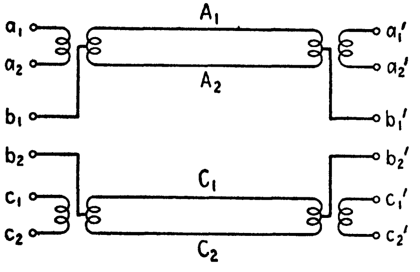


FIG. 9-18. Phantom circuit and side circuits.

Figure 9-18 indicates the arrangement of the phantom group, as the three circuits are called. The circuit connecting terminals a_1a_2 to $a_1'a_2'$ comprises wires A_1 and A_2 with a repeating coil (transformer) at each end. The other circuit, c_1c_2 to $c_1'c_2'$, is carried over wires C_1 and C_2. The phantom channel is from terminals b_1b_2 to terminals $b_1'b_2'$.

If current enters at terminal b_1, it will divide evenly between wires A_1 and A_2; it will not magnetize the repeating-coil winding between these wires, for the upper and lower halves of the repeating-coil winding carry equal current and hence produce equal and opposite magnetomotive force in the core. Consequently, the coil will offer no reactance to the current, and no voltage will appear between terminals a_1 and a_2.

Currents of the phantom circuit will likewise flow in opposite directions through the repeating coil at the distant end of the line. As the core is not magnetized, no voltage appears between terminals a_1' and a_2'. The current is delivered at b_1', passes through the terminal equipment, and returns to the phantom circuit at b_2'. Dividing equally between wires C_1 and C_2, the current returns to terminal b_2. Thus the phantom circuit can theoretically operate without producing crosstalk in either of the side circuits.

It is easy to see that the side circuits will not produce crosstalk in the phantom circuit, assuming they are accurately balanced and transposed. Operation of the a_1a_2 to $a_1'a_2'$ circuit leaves the mid-point of the repeating coil at each end of the A_1A_2 line always at ground potential. (This is what is meant by having the line balanced.) Hence signal currents in that circuit will have no effect on the phantom circuit. Similarly, the phantom is unaffected by current in the C_1C_2 side circuit. In short, all three circuits are independent of each other.

In the construction of open-wire lines using the common ten-pin crossarm, the four outer pins on each end of the arm constitute a phantom group. Other combinations of four adjacent wires are also *phantomed.* The problems of transposing and balancing are severe if the high frequencies of carrier operation are used on phantom groups, and the economic advantages of carrier are such that new line construction is largely without phantom circuits.

In a cable, the two wires of each pair are twisted together and the two pairs are twisted in a *quad.* The many quads in a cable are then spiraled, with alternate layers turning in opposite directions. This complicated pattern gives the equivalent of transposition of circuits.

14. Irregularities. A technique of practical value for locating short circuits, faulty joints, and other irregularities on telephone lines makes use of transmission-line theory. The input impedance to a line is measured at a number of frequencies, using an impedance bridge, and from such measurements the distance to the fault can be determined.

If a telephone line is in good condition and properly terminated, application of voltage at the sending end produces a single traveling wave. There is no reflection at the load, or elsewhere, and input impedance is equal to Z_0.

Any kind of an irregularity in the line, however, will reflect some part of the incident wave, and a reflected wave as well as an incident wave will appear on that part of the line that is between the transmitter and the irregularity. There will be standing waves on the line between the transmitter and the irregularity (see Fig. 9-19) although not between the irregularity and the load.

With an irregularity in the line, input impedance to the line has resonant variations with changing frequency. The measured impedance has maxima and minima, as in Fig. 9-20.

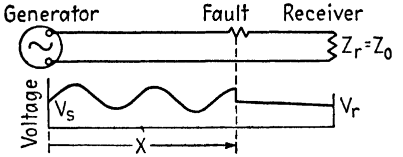


FIG. 9-19. Standing waves produced by partial reflection at a fault.

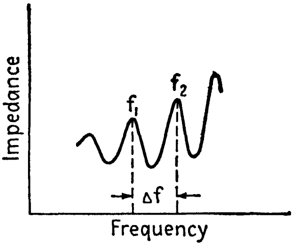


FIG. 9-20. Impedance measured on a line with an irregularity.

It was explained in Sec. 4, Chap. 3, that maxima (or minima) in such a curve as Fig. 9-20 occur each time the value of βx changes by π. (This was discussed for a line without loss, but it applies equally well, although with less sharp maxima and minima, if there is loss.) Call the frequency of one maximum f_1 and that of the next f_2, and the frequency difference between them Δf; this is the change of frequency that increases βx by π. In other words, this is the change of frequency that gives one more half-wavelength in the length x of the line from generator to fault. At frequency f_1, there are n half-wavelengths in the distance x:

$$x = n\left(\frac{1}{2}\lambda_1\right) = \frac{n(\text{velocity})}{2f_1} \tag{9-65}$$

At frequency f_2, there are $n + 1$ half-wavelengths in the same distance:

$$x = (n + 1)\left(\frac{1}{2}\lambda_2\right) = \frac{(n + 1)(\text{velocity})}{2f_2} \tag{9-66}$$

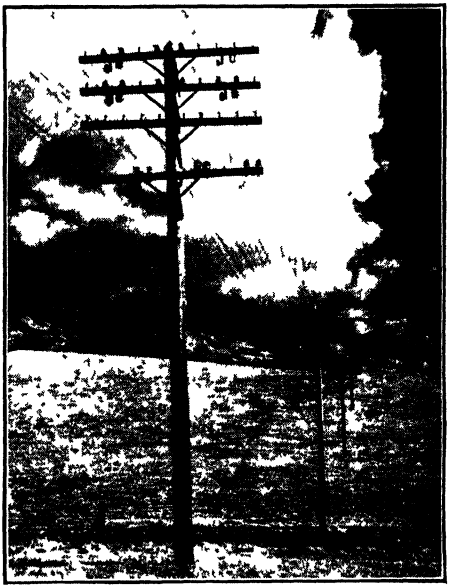
Simultaneous solution gives

$$x = \frac{\text{velocity}}{2\Delta f} \tag{9-67}$$

as the distance from the transmitter to the fault. This solution assumes the velocity is independent of frequency; if the velocities v_1 and v_2 at f_1 and f_2 are appreciably different, the result is

$$x = \frac{v_1 v_2}{2f_2 v_1 - 2f_1 v_2} \tag{9-68}$$

15. American Telephone Practice. The following précis by Cone and Fowler[11] describes the types of construction of long telephone lines in the United States.



An open-wire telephone line. Three upper crossarms show standard arrangement of ten insulator pins with 12-inch spacing for flat phantom groups, side-circuit transpositions are seen on two top arms. Fourth crossarm shows 28-inch spacing of circuits for Type J carrier; the left pair has a point-type transposition on this pole; see Fig. 5-2. (*Courtesy of Bell Telephone Laboratories.*)

Prior to 1930, growth was characterized by the use of voice-frequency cable circuits and by open-wire telephone lines with one and three-channel carrier systems in the 4 to 30 kilocycle range.

In the 30's, first the revival of business brought the numbers of telephones and toll messages to new peaks, and then the beginning of World War II set up a demand resulting in another sharp advance during and since the war years.

It would of course have been possible to resume the construction of the voice frequency cable network which had been halted in 1930. Most fortunately, however, the activity of research and development engineers during the depression years had made ready new carrier systems, known as "J" and "K", providing circuits of greater transmission capabilities and making such effective use of existing open-wire lines and cables, both aerial and sub-surface, that the requirements for copper, lead, poles and rights-of-way were greatly lessened, with corresponding reduction in costs. In addition to these systems, which are described more fully below, new types of transmission facilities have been developed for long-haul routes requiring large numbers of message circuits or television channels utilizing respectively, coaxial cable structures in the "L" carrier system and microwave radio techniques in the "TD" radio-relay system, thus further enlarging the range of choice of instrumentalities.

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[11] From D. I. Cone and C. V. Fowler, "Expanding the Pacific Coast Telephone Network," a conference paper, American Institute of Electrical Engineers, Aug. 24, 1949.

Instrumentalities. The J, K and L carrier systems have much in common. The same type of terminal equipment is used for stacking 12 voice circuits one above another in a frequency block 48 kilocycles wide in the range of 60-108-KC, and for segregating the channels after traversing the line. This equipment determines the transmission band width of a single carrier channel, about 100 to 3500 cycles. Use of the same 12-channel frequency block makes the interconnection of J, K and L circuit groups feasible without separating individual channels, and adds much to the economy and flexibility of these systems. After passing the initial stacking process the systems are no longer alike. Further modulation stages are used in each case to shift each 48-kilocycle block of twelve voice-channels to a frequency range satisfactory for the type of line facility to be used.

The J system is intended for open-wire, providing twelve two-way telephone channels on one pair. Where several J systems are to be operated on the same pole-line, standard open-wire pairs such as are commonly used for voice circuits are generally unsuitable until they are retransposed and respaced to lessen the crosstalk coupling. The fundamental 48 kilocycle block is shifted to the range 36 to 143 kilocycles with one direction of transmission using the lower part of the range and the opposite direction of transmission the upper. Repeaters are placed at intervals of 30 to 100 miles depending upon the gauge of wire used and sleet conditions along the route. A three-channel carrier system, type "C", can operate on the pair along with the J, making a total of sixteen circuits on one pair of wires.

The K system provides twelve telephone channels on two pairs of standard 19-gauge voice-frequency toll cable, one non-loaded pair being used for each direction of transmission. The fundamental 48 KC block is shifted to the 12-60 KC range for transmission over the line. Repeaters are required about every 17 miles. Since the same frequency band is used for the two directions, shielding is necessary to prevent near-end crosstalk and is usually provided by employing pairs in separate lead-sheathed cables. Far-end crosstalk is controlled at certain repeater points by a complicated balancing panel that couples every K pair to every other K pair in the same cable, the coupling coils and condensers being adjustable to neutralize the cross-talk.

The L-1 coaxial cable carrier system provides up to 600 telephone circuits over two coaxial conductors, one for each direction of transmission. The present standard coaxial is a tube of three-eighths inch inside diameter with a central wire of No. 10 AWG held in place by polyethylene discs, about one inch apart. As many as 8 coaxials are placed in one cable sheath along with ordinary paper-insulated cable pairs. Five of the fundamental 48 KC blocks of 12 channels are stacked on top of one another in the range of 312-552 KC to provide a 60-channel supergroup. As many super-groups as required up to a total of ten can be stacked using the frequency range of 64-3100 KC for transmitting on the line. Provision is made for dropping groups of circuits in multiples of 12 for service to intermediate points along the route. Repeaters are needed at 7 to 8 mile intervals. Main repeating points with power supply are required about every 150 miles, and power is furnished to the auxiliary repeater stations via the coaxial inner conductors. For television transmission the video signal is shifted to the 300 KC-3100 KC band, and appropriate equalizers are added at repeater points.

The problem of separating the many frequency bands involved in these systems is solved by the use of channel-separation filters in which quartz (or synthetic) crystals are used as reactive elements. The large number of high-gain repeaters involved makes it necessary to have amplifiers of very high stability and low modulation, and these are obtained through the use of negative feedback.

Maximum power amplification per repeater is some 40,000 times (46 db) for the J, 3,000,000 times (65 db) for the K and 300,000 times (55 db) for the L-1, and is utilized to compensate for the maximum loss in one repeater section. Each system provides for automatic adjustment of repeater gains to correspond with changes in line losses brought about by temperature variations and in the case of the J system by moisture and ice on the wires as well.

Transmitted power levels are kept as low as possible consistent with favorable signal to noise ratios. They vary from about 50 milliwatts per channel in the case of the J system where open-wire noise may be appreciable down to 0.1 milliwatt in the case of the L-1 system where highly shielded coaxials are employed. In the case of the K system, inductance coils prevent extraneous energy on voice-frequency circuits in the same sheaths from causing noise.

On the telephone channels derived from any of these systems 18-channel voice frequency telegraph systems may be operated.

The broad-band carrier systems just described have provided the great bulk of added toll circuit growth, for the larger groups of circuits and the longer routes. For service to mobile stations such as automobiles and ships, and for locations where wire or cable plant is difficult or impracticable, radio links are in use, ranging in number of channels from one to sixteen, in carrier frequency from 2 to 4000 megacycles, and in length from a few miles to thousands of miles, utilizing amplitude, frequency or pulse modulation techniques.

Microwave radio-relay systems are designed to handle many hundreds of telephone message channels or a television signal on one radio circuit.

The TD type radio-relay system operates on a point-to-point basis in the 3700-4200 megacycle range. It uses the same type of terminal equipment as the L carrier to stack telephone channels. A 70-megacycle intermediate carrier is then frequency-modulated by the combined block of L carrier frequencies. Relay-stations are so located, usually some tens of miles apart, depending on the terrain, as to have clear lines of sight between the sharply beamed lens antennas fed by transmitting amplifiers delivering about 0.5 watt. The normal loss of the radio path for 35 miles of this system is about 64 db, comparable to that of a long K-cable section and about 9 db more than the loss in a 7.9 mile repeater section of coaxial cable for the L-1 system as previously described. Each antenna may be used for as many as six one-way radio channels on a route, each such channel accommodating an L-1 carrier system or one television channel.

In addition to the savings in money and materials made possible by carrier systems, their higher speed of transmission compared with the voice frequency loaded cables of the earlier art is an important advantage in lessening the effects of echoes. The importance of echoes in any two-way circuit increases as the time for the voice to travel over the facility increases. The carrier systems have speeds over 100,000 miles per second and can be about five times as long as the cable circuits, which have speeds of 20,000 miles per second, for the same amount of echo impairment.

As an aid in securing satisfactory signal-to-noise ratios, devices known as compandors, a name coined from compressor and expandor, have been used in connection with the open wire "J" system. The compandor automatically introduces more gain at the transmitting end of a given voice-channel and more loss at the receiving end for low level signals than for high level ones, but with no change in overall transmission loss of the channel. This results in an improvement in noise and crosstalk performance of some 20 to 25 db for silent intervals and low speech volumes, where interference is particularly objectionable.

PROBLEMS

9-1. For the telephone line of Fig. 5-2, what are the values of characteristic impedance at the highest and lowest frequencies in the "present general average" telephone frequency band? What would you consider the best terminal impedance for such a line for ordinary speech transmission? Explain.

9-2. Using the power ratios given in the text (page 188) for 1, 3, 7, and 10 db, compute, by simple combination of these, approximate power ratios for 2, 4, 5, 6, 8, and 9 db. Compare these with exact values from Table 9-1.

9-3. Using the power ratios given in the text (page 188), compute, by simple combination of these, approximate power ratios for 13, 14, 25, 47, and 58 db.

9-4. Using values from Table 9-1, find the power ratios for 13, 25, 47, 58, 27.4, -11, -36, and -82 db. (Example: 13 db = 10 + 3 db; power ratio = $10.00 \times 1.99 = 19.9$.)

9-5. Using a table of logarithms, compute decibels corresponding to power ratios of 2.00, 4.53, 0.667, 1,492, and power ratios corresponding to decibel values of 1.00, 14.52, -0.621, 57.03.

9-6. Prove by differentiating Eq. 9-17 that $R_a = R_b$ is the condition for maximum power transfer. (It is previously stipulated that $X_a = -X_b$.)

9-7. A telephone amplifier (Type K carrier) has an input transformer that matches 140 ohms to 30,000 ohms. What is the turn ratio? It has an output transformer that matches 3,500 to 140 ohms. What is the turn ratio?

9-8. A telephone amplifier (44-type repeater) has an input transformer that matches 700 ohms to 20,000 ohms, and an output transformer that matches 6,000 ohms to 700 ohms. Find the turn ratio of each.

9-9. Design an L-type network of reactive elements to match a resistance R_1 (Fig. 9-7) of 500 ohms to a resistance R_2 of 1,000 ohms. Frequency is 20,000 cycles per second.

9-10. Design an L-type reactive network to match a 300-ohm circuit to a 750-ohm circuit (both purely resistive). Frequency is 35,000 cycles per second.

9-11. The network of Prob. 9 is connected to its 1,000-ohm output resistance. What is the input impedance at 20,000 cycles? at 23,000 cycles?

9-12. The network of Prob. 10 is connected to a 750-ohm output resistance. What is the input impedance at 35,000 cycles and at 32,000 cycles?

9-13. Find the phase shift in the matching network of Prob. 9-9.

9-14. Find the phase shift in the matching network of Prob. 9-10.

9-15. Design a T network to provide the match required in Prob. 9-9 and to give a phase advance of 45 degrees.

9-16. Design a Π network to provide the match required in Prob. 9-10 and to give a phase retardation of 30 degrees.

9-17. How much shunt conductance (per mile) would have to be added to the open-wire telephone line of Table 9-2 to make it a distortionless line? How much would the attenuation be increased in decibels per mile at 1,000 cycles? Explain why this is not practical.

9-18. How much inductive loading (per mile) would be required to make the nonloaded telephone cable of Table 9-2 a distortionless line? Compare the advantage gained by moderate loading, as in the loaded cable of Table 9-2, with the advantage that could be gained by loading the line to be distortionless (a comparison can be made in terms of decibel gain per millihenry of loading).

9-19. Values of l, α, β, Z_0, velocity, and decibels per mile for the loaded cable of Table 9-2 were computed by assuming that the inductance and resistance added by loading coils were uniformly distributed. Check these values, which are for 1,000 cycles per second.

9-20. Continuing Prob. 9-19, compute more precise values by the method of Fig. 9-11. Compute cutoff frequency.

9-21. Compute values as for Prob. 9-19, but at 3,000 cycles.

9-22. Compute values as for Prob. 9-20, but at 3,000 cycles.

9-23. Design circuits to be inverse (in the meaning of Eq. 9-64) to each of the four circuits of Fig. 9-8. (Numerical values of elements are *not* required.)

9-24. Z_0 of a telephone line is 650 ohms. A faulty splice introduces 200 ohms resistance, as in Fig. 9-19. Sketch voltage and current distribution along the line, on both sides of the fault, including at least $\frac{1}{2}$ wavelength from the fault toward the source. Received voltage is 10 volts at 1,000 cycles per second. (Precise computation is not wanted. Maxima and minima should be approximately correct in position and relative magnitude. Show some, but not much, attenuation.)

9-25. Impedance maxima in Fig. 9-20 are measured at 1,730, 2,460, and 3,200 cycles frequency. The circuit is a loaded cable, 6,000 feet between loads, velocity of propagation 17,640 loads per second (20,000 miles per second equivalent). Find the location of the fault.

CHAPTER 10

INTRODUCTION TO FILTERS

1. Purpose. The value of an electric-wave filter for separating electric currents of one frequency from those of another is obvious. In the last chapter, for example, reference was made to carrier-current telephone systems in which a number of conversations are transmitted on one pair of wires; each signal channel utilizes a distinct frequency band, and the signals are separated at the receiving end of the line by means of filters. Filters have innumerable applications in communication work, including both audio-frequency applications and uses at higher video and radio frequencies, although they have perhaps been most highly developed for long-distance telephone systems.

Electric-power engineering also requires the application of the principles of filters, not only when high-frequency carrier-current signals are superimposed on the conductors of power lines for the purpose of communication or automatic relaying, but also when harmonic frequencies are to be separated from the fundamental power frequency. Indeed, filters of a sort are used with power rectifiers to give pure direct current. Rectifier filters are designed from a specialized point of view, however, as they should pass only the zero-frequency component of current and stop all other frequency components.

The ideal communication filter would pass a selected band of frequencies with no attenuation or distortion of the signal, while completely preventing the passage of frequencies outside that selected band. This ideal cannot be attained. The basic problem of filter design is to approach the ideal as closely as possible.

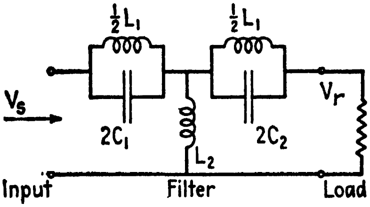

FIG. 10-1. A filter circuit. $L_1 = 0.2085$, $L_2 = 0.0930$, $C_1 = 0.1890 \cdot 10^{-6}$, $R = 700$.

It is not absolutely necessary to develop any special mathematical approach to analyze the operation of a filter circuit. The familiar principles of circuit theory are applicable. However, they are extremely laborious, as will be seen from a single illustration.

2. Elementary Computation. Consider the filter network of Fig. 10-1 connected between a source of energy and a purely resistive load of 700 ohms. This network, connected to such a load, is designed to pass all frequencies above 1,000 cycles per second, and to stop all lower frequencies: it is a high-pass filter. Let us compute its actual operation, and compare with the ideal proposed above.

The voltage across the load, which is also the voltage at the receiving end of the filter, will be called V_r. The input voltage is V_s. Since all the elements of the network are known, V_s can be expressed in terms of the frequency ($\omega = 2\pi f$) and V_r as follows:

$$\frac{V_s}{V_r} = 1 + \frac{1}{2}\frac{L_1/L_2}{1 - \omega^2 L_1 C_1} + j\frac{1}{R}\left[\frac{\omega L_1}{1 - \omega^2 L_1 C_1} + \left(\frac{1}{2}\frac{\omega L_1}{1 - \omega^2 L_1 C_1}\right)^2 \frac{1}{\omega L_2}\right] \qquad (10\text{-}1)$$

By substituting numerical values into this expression, we find (after a few hours of computation) the results shown in Fig. 10-2.

In Fig. 10-2*a* it is seen that V_r is practically the same as V_s for all frequencies above 1,000 cycles; this part of the operation of the filter is very satisfactory. At frequencies below 1,000 cycles, the operation is not uniformly good. At just 800 cycles, no signal reaches the load, but at other frequencies there is more or less voltage across the receiving-end terminals.

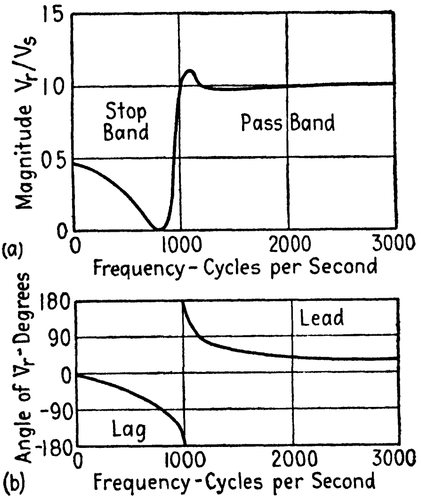

FIG. 10-2. Voltage at load through the filter of Fig. 10-1.

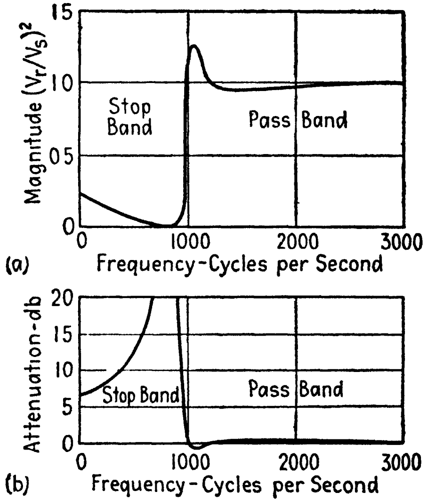

FIG. 10-3. Power to the load through the filter of Fig. 10-1.

The primary interest is usually in power delivered to the load, and since the load, whatever its actual nature, is usually best represented by a constant resistance, the received power varies as the square of the applied voltage. Hence we plot in Fig. 10-3*a* the magnitude of $(V_r/V_s)^2$ and interpret this quantity as *received power*; to be more explicit, this is the ratio of the power received by the load with the filter in the circuit to the power that would be received by the load if the filter were removed and the input voltage, unchanged in magnitude, were then applied directly to the load terminals.

When power to the load is considered, the filter characteristics look reasonably good. The maximum power transmitted in the stop band (except in the immediate region of the cutoff frequency of 1,000 cycles) is only about two-tenths of the power that would be transmitted if there were no filter. We shall design better filters, however, a little later in the chapter.

It will be remembered that the decibel is a unit of power ratio. It is interesting to plot in decibels the power ratio of Fig. 10-3*a*, as is done in 10-3*b*. The power ratio is here labeled "attenuation" in usual filter terminology, but this does not mean that power is lost in transmitting a signal through the filter. The word *attenuation* has not quite the same meaning when related to filters that it has in connection with transmission lines. On a transmission line, attenuation refers to the energy loss of a pure traveling wave that results from resistance and leakage; in a filter, attenuation expresses the amount of reduction of signal strength that results from the presence of the filter.

When a filter prevents power from reaching a load, it does not dissipate that power. Note that the filter circuit of Fig. 10-1 could not dissipate any power, for all its circuit elements are assumed to be purely reactive; yet it prevents power from reaching the load. It does this by rejecting power at the filter input terminals if the applied frequency is in the stop band. Assuming that all filter elements are reactive (and this is the usual basis of design), *the amount of power entering the filter is identical with the amount of power received by the load.*

Power is not admitted by the filter (and hence power is not received by the load) at frequencies in the stop band primarily because *the input impedance to the filter is reactive in the stop band.* That is, the network of Fig. 10-1, comprising both filter and load, as seen looking in at the input terminals, appears almost purely reactive to stop-band frequencies. To pass-band frequencies, however, it appears almost purely resistive; thus the filter receives power and transmits it to the load: *the input impedance to the filter is resistive in the pass band.*

At some stop-band frequencies, the input impedance is very high as well as being almost purely reactive. For this reason, also, little power is admitted by the filter. However, in other parts of the stop band the input impedance is quite low, but being still reactive the power admitted is small and the so-called attenuation remains high.

Some authors distinguish between the attenuation that results from the absorption or dissipation of power, and the attenuation that is a measure of rejected power (as with a filter), by referring to the latter as *reactive attenuation*. It is this reactive attenuation that is shown in Fig. 10-3*b*.

Input impedance to the filter circuit of Fig. 10-1 can be computed by familiar methods, combining elements in parallel and series combinations, as follows:

$$Z_{in} = \frac{\left(R + j\frac{1}{2}\frac{\omega L_1}{1 - \omega^2 L_1 C_1}\right)\left(j\omega L_2\right)}{R + j\left(\frac{1}{2}\frac{\omega L_1}{1 - \omega^2 L_1 C_1} + \omega L_2\right)} + j\frac{1}{2}\frac{\omega L_1}{1 - \omega^2 L_1 C_1} \qquad (10\text{-}2)$$

Substituting numerical values into this expression one obtains (after several more hours of calculation) the results shown in Fig. 10-4. The input impedance is not radically different from the load resistance, 700 ohms, through the pass band. In the stop band it rises rapidly, becoming infinite at 800 cycles (this assumes ideally perfect reactive filter elements) and then falling to zero at zero frequency. The angle of the input impedance changes with truly remarkable rapidity at the cutoff frequency of 1,000 cycles from nearly zero degrees to nearly −90 degrees; then at 800 cycles, which is the frequency of parallel resonance of the arms of the filter circuit, it changes abruptly to +90 degrees. The essential fact is that the input impedance is resistive in the pass band and reactive in the stop band, thereby controlling the entrance of power to the filter.

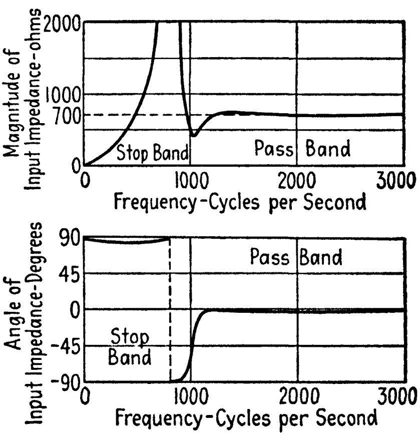

FIG. 10-4. Input impedance to filter of Fig. 10-1.

This computation using the elementary formulas for combining impedances in parallel and series, which might (by comparison with more skillful approaches) be called the brute-force method, can obviously be applied to filter circuits if one wishes to do so. It is quite laborious, but this is not the chief objection. Its great fault is that it fails to show general relations, or to suggest principles of design. How could one guess that the circuit of Fig. 10-1 would have properties even approximating those of a good filter? How could one guess its cutoff frequency? Is there any indication that it would match a 700-ohm resistance load? The arms give resonance at 800 cycles, indeed, so clearly this is a frequency of infinite attenuation, but this is only one of many factors involved in filter operation.

The study of filters can be approached in a systematic and reasonably simple manner through either network theory or transmission-line theory. The network-theory approach is the more general, but the development through transmission-line theory is a natural outgrowth of the preceding chapters of this book.

3. The Filter as an Artificial Line. In Chap. 9, it was seen that a loaded telephone line has relatively low attenuation for low frequencies, and an attenuation that is many times as great for frequencies above a critical frequency known as cutoff. The loaded line is therefore, to this extent at least, a filter.

Conversely, as was shown in Chap. 8, one can determine the parameters that will make a line with distributed constants equivalent to any given symmetrical T or Π section. It may not be possible actually to construct a line with the parameters so determined, but that need not deter our consideration of such a line—we can compute its characteristic impedance, its attenuation, and phase shift.

We could find, for example, the parameters of a line equivalent to the filter section of Fig. 10-1. Such a line would have high attenuation at low frequencies but practically no attenuation above 1,000 cycles. It would have a characteristic impedance of 700 ohms at frequencies above 1,000 cycles, to match the load; below 1,000 cycles the characteristic impedance would be reactive.

To be more general, let us consider the relations that appear when any T or Π section is replaced by a transmission line. General equations were developed in Chap. 8, and may be taken from Table 8-1, but to be consistent with usual filter notation the following changes in symbols will be made. Series and shunt impedances will be called Z_1 and Z_2 as indicated

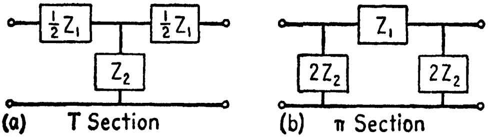

Fig. 10-5. Filter notation.

in Fig. 10-5 ($Z_1 = Z$ of Chap. 8; $Z_2 = 1/Y$ of Chap. 8), and α and β will be attenuation (in nepers) and phase shift (in radians) per *filter section* (in Chap. 8 these were attenuation and phase shift per unit length of line).

With these modifications of symbols, the equations of Table 8-1 will be used to find α and β of an equivalent line. For a line that is equivalent to either the T or the Π section of Fig. 10-5, we first find B or γ:

$$\cos B = 1 + \frac{1}{2}\frac{Z_1}{Z_2} \qquad B = \cos^{-1}\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right) \tag{10-3}$$

or

$$\cosh \gamma = 1 + \frac{1}{2}\frac{Z_1}{Z_2} \qquad \gamma = \cosh^{-1}\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right) \tag{10-4}$$

The phase shift β and the attenuation α are known from the real and imaginary components of B:

$$B = \beta - j\alpha \tag{10-5}$$

or, if γ is used instead of B,

$$\gamma = \alpha + j\beta \tag{10-6}$$

(As in previous chapters, either trigonometric or hyperbolic functions can be used, as preferred.)

Thus we have a means of computing attenuation and phase shift in the filter section, for since α and β give attenuation and phase shift in the equivalent line, they also give attenuation and phase shift in the filter section.

We next find the characteristic impedance of the equivalent line, and the quantity so determined may be called, for brevity, the characteristic impedance[1] of the filter section. From the equations of Table 8-1, the characteristic impedance of a T section is

$$Z_{0T} = \sqrt{Z_1 Z_2 + \tfrac{1}{4}Z_1^2} \tag{10-7}$$

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[1] In network theory, the term *image impedance* of a filter section is preferred to *characteristic impedance*, the latter term being reserved for a line with distributed constants. See "American Standard Definitions of Electrical Terms" (Ref. 1). (In the present discussion, where the term *characteristic impedance* is applied to a filter section, it is to be understood as being the characteristic impedance of the equivalent line.) Also, α and β are the *image attenuation constant* and *image phase constant* of network-theory nomenclature, the real and imaginary parts, respectively, of the *transfer constant* γ.

and for a Π section the characteristic impedance is

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2 + \frac{1}{4} Z_1^2}} = \frac{Z_1 Z_2}{Z_{0T}} \tag{10-8}$$

To understand the meaning of the characteristic impedance of a filter section, merely consider it to be the characteristic impedance of the equivalent transmission line, a line with identical terminal behavior. As a line, to avoid resonance effects, must be terminated in a load with impedance equal to the characteristic impedance of the line, so, a filter section should be terminated in its characteristic (or image) impedance. Two or more lines can be connected in cascade[2] with no wave reflection at the junctions, and hence no resonance effects, if the lines have the same characteristic impedance. Two or more filter sections can likewise be connected in cascade without producing undesired resonance effects if they have the same characteristic (or image) impedance.

4. Transmission and Attenuation Bands. When we suppose a line to be substituted for a filter section, we must understand that the parameters (α, β, B, γ, and Z_0) of the equivalent line change with frequency. This is true because Z_1 and Z_2 of the filter section are functions of frequency, and these appear in Eqs. 10-3, 10-7, and 10-8. Much of the behavior of the filter section is deduced from the manner in which the parameters vary with frequency.

Thus, from Eq. 10-3, $\cos B = \cos(\beta - j\alpha) = 1 + \frac{1}{2}Z_1/Z_2$. If the value of $1 + \frac{1}{2}Z_1/Z_2$ is such that B is purely real, it follows that $\alpha = 0$, and there is no attenuation, but phase shift only. If, on the other hand, B turns out to be imaginary, there is attenuation but no phase shift.

If, in the above equation for $\cos B$, Z_1 and Z_2 are reactances without loss, $\cos B$ is a real quantity; it may have any magnitude, and may be either positive or negative. If $\cos B$ is between -1 and $+1$, B is real, but if $\cos B$ is greater than $+1$ or less than -1 no real value of B will suffice.

Reference to the chart of Fig. 4-4 gives the following information:

1. If $1 + \frac{1}{2}Z_1/Z_2$ is between -1 and $+1$ (in Fig. 4-4, $S < 1$ and σ is either 0 or ±180°), B is real and there is phase shift but no attenuation.

2. If $1 + \frac{1}{2}Z_1/Z_2$ is greater than $+1$ ($S > 1$ and $\sigma = 0$ in Fig. 4-4), there is attenuation, and phase shift is zero.

3. If $1 + \frac{1}{2}Z_1/Z_2$ is less than -1 ($S > 1$ and $\sigma = \pm 180°$ in Fig. 4-4), there is attenuation, and phase shift is 180 degrees (π radians).

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[2] As another item of nomenclature, four-terminal networks such as those of Fig. 10-5, when connected together, with the output terminals of one joined to the input terminals of the next, are more properly referred to as being in *cascade* than in series. This terminology will be used hereafter.

The same conclusion is, of course, reached from consideration of Eq. 10-4. If γ is real, there is attenuation; if it is purely imaginary, there is none. If γ is imaginary, $\cosh \gamma$ lies between -1 and $+1$, so if $1 + \frac{1}{2}Z_1/Z_2$ is between -1 and $+1$ there is no attenuation. If, on the other hand, the quantity $1 + \frac{1}{2}Z_1/Z_2$ is less than -1 or greater than $+1$, there is attenuation, and phase shift is either zero or 180 degrees.

The frequency range in which there is attenuation is called the *attenuation band* or, for brevity, the *stop band* of the circuit. (The latter term, although common, is somewhat misleading; a good filter gives large attenuation in the *stop band*, but no band of frequencies is completely stopped.) The frequency range in which there is no attenuation is called the *transmission band* or *pass band.*

If the frequency is such that $1 + \frac{1}{2}Z_1/Z_2$ is between -1 and $+1$, that frequency is in the pass band. This relation can be more usefully expressed in other words: for frequencies in the pass band, $\frac{1}{2}Z_1/Z_2$ lies between -2 and 0. The pass band, then, can be defined by the condition that

$$\frac{Z_1}{4Z_2} \text{ lies between } -1 \text{ and } 0 \tag{10-9}$$

Neglecting loss, and letting $Z_1 = jX_1$ and $Z_2 = jX_2$, the pass band is given by

$$\frac{X_1}{4X_2} \text{ lies between } -1 \text{ and } 0 \tag{10-9a}$$

This leads to the following criterion for any Π or T network section: *in the pass band, X_1 and X_2 must be of opposite sign, and $4X_2$ must be numerically greater than X_1.*

It is clear that a T or Π network may have no pass band. An example is one in which Z_1 and Z_2 are both inductances. A circuit with no pass band is, of course, not a filter. We must search for networks that have properties similar to those postulated for the ideal filter.

Both Z_1 and Z_2 can be combinations of inductances and capacitances in series and parallel arrangements. There is literally no limit to the possible complexity of a T or Π section. For practical use, however, a filter with a small number of elements is highly desirable. Let us, therefore, give first consideration to the simplest types of filter section.

5. Inverse-network Filters. It has been mentioned that an artificial transmission line is a filter and, indeed, it is one of the simplest possible. A T section is shown in Fig. 10-6*a*, with

$$Z_1 = j\omega L \qquad X_1 = \omega L \tag{10-10}$$

$$Z_2 = \frac{1}{j\omega C} \qquad X_2 = -\frac{1}{\omega C} \tag{10-11}$$

The first requirement of the pass band, that X_1 and X_2 must be of opposite sign, is met at all frequencies. The second is that $4X_2$ must be numerically greater than X_1:

$$|\,4X_2\,| > |\,X_1\,| \tag{10-12}$$

This requires that in the pass band

$$\frac{4}{\omega C} > \omega L \tag{10-13}$$

$$\omega = 2\pi f < \frac{2}{\sqrt{LC}} \tag{10-14}$$

$$f < \frac{1}{\pi\sqrt{LC}} \tag{10-15}$$

The circuit is therefore a low-pass filter and the cutoff frequency is $f_c = 1/\pi\sqrt{LC}$.

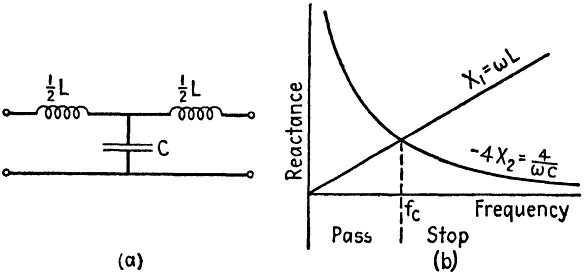


FIG. 10-6. Low-pass inverse filter.

The pass band of this filter section is located graphically by Fig. 10-6*b*. X_1 and $-4X_2$ are both plotted as functions of frequency, and the point at which they cross is the cutoff frequency. All lower frequencies satisfy the criterion of Eq. 10-9, and therefore constitute the pass band.

This kind of graphic analysis gives a simple indication of the pass and

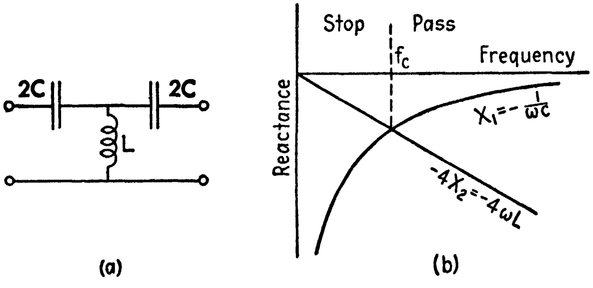


FIG. 10-7. High-pass inverse filter.

stop bands of a T or Π section. As another example, consider the T section of Fig. 10-7*a*. The reactance terms X_1 and $-4X_2$ are plotted in *b* of the figure. They have the same sign for all frequencies, and $-4X_2$ is numerically the larger for frequencies greater than that at which the curves cross. Hence, the section is a high-pass filter and its cutoff frequency is found as follows:

$$4\omega_c L = \frac{1}{\omega_c C} \tag{10-16}$$

$$\omega_c^{\;2} = \frac{1}{4LC} \tag{10-17}$$

$$f_c = \frac{1}{4\pi\sqrt{LC}} \tag{10-18}$$

The filters of Figs. 10-6 and 10-7 are called inverse-network or constant-*k* filters because the product of the impedances is independent of frequency:

$$Z_1 Z_2 = k^2 \tag{10-19}$$

For both high-pass and low-pass sections, $k = \sqrt{L/C}$.

Constant-*k* filters have the great practical advantage of simplicity. They have two disadvantages, however, that must be overcome before they can be used in practical filter design.

6. Amount of Attenuation. One disadvantage of the constant-*k* filter is that is does not give as much attenuation in the stop band as is ordinarily wanted, particularly at frequencies near cutoff.

Attenuation produced by any T or Π filter in the stop band is found from Eq. 10-4:

$$\cosh\gamma = 1 + \frac{1}{2}\frac{Z_1}{Z_2} \tag{10-20}$$

It will be recalled that there is attenuation if the right-hand member is greater than +1 or if it is less than −1. If $1 + \frac{1}{2}Z_1/Z_2$ is greater than +1, γ is real. That is, from Eqs. 10-6 and 10-20,

$$\gamma = \alpha + j\beta = \cosh^{-1}\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right) \tag{10-21}$$

and when γ is a real quantity it follows that

$$\alpha = \cosh^{-1}\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right) \tag{10-21a}$$

and

$$\beta = 0$$

If, on the other hand, $1 + \frac{1}{2}Z_1/Z_2$ is less than -1, it is more convenient to write (using Formula 41 of Table II)

$$\cosh(\gamma + j\pi) = -\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right) \tag{10-22}$$

From this,[3]

$$\alpha = \cosh^{-1}\left[-\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right)\right] \tag{10-23}$$

$$\beta = -\pi \tag{10-24}$$

As a general expression for either part of the attenuation band, we can write

$$\alpha = \cosh^{-1}\left|1 + \frac{1}{2}\frac{Z_1}{Z_2}\right| \tag{10-25}$$

This gives attenuation in nepers in the section of filter. For decibels, multiply by 8.686.

Applying the general equation 10-25 to the low-pass filter of Fig. 10-6, attenuation = $\cosh^{-1}(\frac{1}{2}\omega^2LC - 1)$. Numerical substitution gives the attenuation characteristic of this low-pass constant-k filter section as shown in Fig. 10-8*a*. Since $1 + \frac{1}{2}Z_1/Z_2 = 1 - \frac{1}{2}\omega^2LC$ and, in the whole attenuation band, this is less than -1, there is a phase shift of 180 degrees in the filter; that is, the output voltage is opposite in phase to the input voltage.

Similarly, the attenuation of the high-pass filter section of Fig. 10-7 is $\cosh^{-1}[1/(2\omega^2LC) - 1]$. The characteristic curve of attenuation is shown in Fig. 10-8*b*.

Cutoff is not sharp with either of these constant-k filter sections; that is, attenuation does not rise rapidly just beyond cutoff. Attenuation is 20 decibels (indicating a reduction of power to 1/100) at about twice the cutoff frequency in the low-pass filter, or about half the cutoff frequency in the high-pass filter. This is not good enough. A good quality of the

---

[3] Since

$$\cosh\gamma = -M$$

where M is a real, positive quantity greater than 1,

$$\cosh(\gamma + j\pi) = M$$

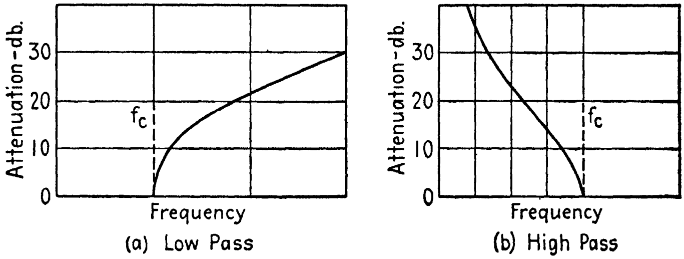
$$\gamma + j\pi = \alpha + j\beta + j\pi = \cosh^{-1} M$$

which is purely real. Therefore

$$\alpha = \cosh^{-1} M$$

$$\beta = -\pi$$

constant-*k* section, however, is that attenuation continues to increase through the entire stop band, and becomes infinite at infinite frequency for the low-pass filter, and infinite at zero frequency for the high-pass filter. This is such a desirable quality that most practical filters include a constant-*k* section to obtain this characteristic.



(a) Low Pass (b) High Pass

FIG. 10-8. Attenuation characteristics of constant-*k* filter sections.

7. Image Impedance. The other principal disadvantage of the constant-*k* filter section is that its image impedance (the characteristic impedance of the line to which it is equivalent) is not constant, but varies markedly with frequency. The result is that the impedance of the filter section fails to match the impedance of its load and the impedance of the source from which it is supplied.

It is not sufficient that a filter match the load and the source at some one frequency. A filter section must match its load and source at all pass-band frequencies. If it does not, power is rejected at the mismatch, and there is attenuation (reactive attenuation) in the pass band.

In terms of the equivalent transmission line, if the characteristic impedance does not match the load impedance there is reflection at the load terminals; power is rejected; a reflected wave is sent back along the line and resonance effects are produced at the sending end. Since the terminal behavior of the filter section is equivalent to that of the line, the same statement may be made about an improperly matched filter section (except, of course, as regards traveling waves, which perhaps can be imagined in a filter though they do not physically exist).

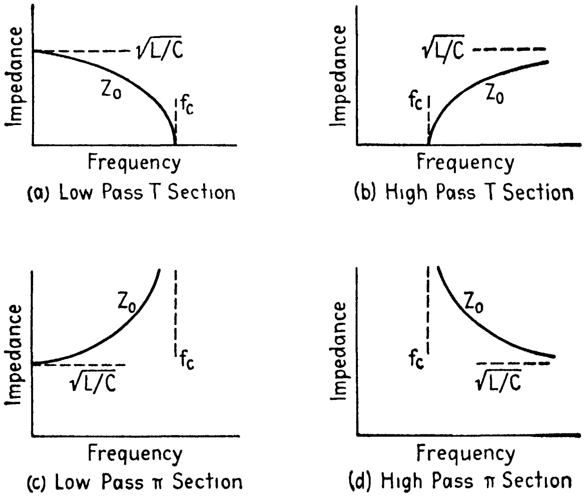
A filter must transmit power freely to the load when a frequency in the pass band is applied. The basic purpose of a filter, which is to be automatically selective, would be lost if the load failed to provide proper termination of the filter section for all pass-band frequencies. The load applied to a filter is most commonly a constant pure resistance. Consequently, when we find that the characteristic impedance (image impedance) of a constant-*k* section is highly variable with frequency through the pass band, it raises a difficult question of impedance matching.

The characteristic impedance of a line equivalent to a T section was shown by Eq. 10-7 to be

$$Z_{0T} = \sqrt{Z_1Z_2 + \tfrac{1}{4}Z_1^2} \tag{10-26}$$

Substituting the appropriate inductances and capacitances of the low-pass T-section filter of Fig. 10-6*a*, it is found that Z_0 varies from $\sqrt{L/C}$ at zero frequency to zero at f_c. The curve is shown in Fig. 10-9*a*.

The high-pass T-section filter of Fig. 10-7*a* has a characteristic impedance that varies in a similar manner through the pass band, as shown in Fig. 10-9*b*. It has a value of $\sqrt{L/C}$ at infinite frequency, and likewise approaches zero at the cutoff frequency.



(a) Low Pass T Section (b) High Pass T Section

(c) Low Pass π Section (d) High Pass π Section

Fig. 10-9. Image (characteristic) impedance in the pass band of constant-*k* and *m*-derived filter sections.

Π sections are different, but no better. The impedance of both low-pass and high-pass types increases without limit as the cutoff frequency is approached. These appear in *c* and *d* of the figure. The problem of providing a satisfactory match for any of these sections must be postponed for consideration in Sec. 10.

Characteristic impedance in the stop band has not yet been discussed. In the stop band, it is not necessary to have the filter impedance match the load or the source impedance. Indeed, it is better if the match is poor. A poor match results in extra attenuation, and in the stop band attenuation is helpful.

It is highly significant that in the stop band the characteristic impedance is purely reactive. Consider Eq. 10-26 for instance. For a filter section without loss, it may be rewritten

$$Z_{0T} = \sqrt{-X_1X_2\left(1 + \frac{X_1}{4X_2}\right)} \tag{10-27}$$

Using the criterion of Eq. 10-9: for stop-band frequencies, either X_1 and X_2 are of the same sign, in which case Z_{0T} is imaginary; or, if they are of opposite sign, $X_1/4X_2$ is less than -1, in which case also Z_{0T} is imaginary.

A similar analysis of the characteristic impedance of a Π section leads to the same conclusion. Hence it is a general conclusion, applicable to any lossless T or Π section, that *for any frequency in the attenuation band, the image (characteristic) impedance of the section is reactive.*

8. Composite Filters. In exploring the good and bad qualities of the constant-*k* filter, we found that it has a desirably high attenuation at frequencies far beyond cutoff, but not enough attenuation near the cutoff frequency. One naturally thinks next of the possibility of putting in cascade with a constant-*k* filter section another filter section of some other kind which will be better near cutoff, although it may be less satisfactory at the extreme limit of the stop band.

Speaking specifically of a low-pass filter, a section with high attenuation just above cutoff, in cascade with a constant-*k* section that has high attenuation at very high frequencies, will together give high attenuation throughout the entire stop band. Total attenuation is the sum of the attenuations of the cascaded sections.

We are immediately faced with the question: Is it permissible to connect two or more filter sections in cascade, or would one destroy the good characteristics of the other? Here the transmission-line representation is specially helpful. Two or more transmission lines can be connected in cascade, and traveling waves will pass freely from one to the next without reflection or any resonant effects, if the lines that are connected together have equal characteristic impedance.

Since the filter section is entirely equivalent to the transmission line as regards terminal conditions, *two or more sections can be connected in cascade, and signals will pass freely from one to the next, if the sections have equal characteristic (image) impedance.*

Plainly, if another section is to be connected in cascade with a constant-*k* section, it must have the same characteristic impedance. Speaking first of a T section, the characteristic impedance is known from Eq. 10-7 to be

$$Z_{0T} = \sqrt{Z_1Z_2 + \tfrac{1}{4}Z_1^2} \tag{10-28}$$

It follows that any two T-network sections can be connected in cascade if they have identical expressions for $Z_1Z_2 + \frac{1}{4}Z_1^2$. To explore this interesting possibility, let us start with the constant-k section as *prototype* (or first model) and see what other sections can be cascaded with it. Any T section, with impedances Z_1 and Z_2, has the same characteristic impedance as the constant-k prototype, with impedances that may be distinguished as Z_{1k} and Z_{2k}, if

$$Z_1Z_2 + \tfrac{1}{4}Z_1^2 = Z_{1k}Z_{2k} + \tfrac{1}{4}Z_{1k}^2 \tag{10-29}$$

Z_1 may be arbitrarily chosen, and related to Z_{1k} by

$$Z_1 = mZ_{1k} \tag{10-30}$$

Substitution into Eq. 10-29 will then eliminate Z_1 and allow us to solve for Z_2:

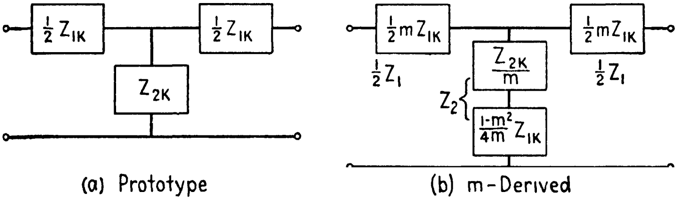
$$Z_2 = \frac{Z_{2k}}{m} + Z_{1k}\frac{1 - m^2}{4m} \tag{10-31}$$

Hence any T section that has Z_1 related to the prototype Z_{1k} by Eq. 10-30, and with Z_2 related to the prototype Z_{2k} by Eq. 10-31, has the same characteristic impedance as the prototype (constant-k) section at all frequencies, and any such sections may be connected in cascade with the prototype or with each other. Such sections are called *m-derived sections.*

There can also be m-derived Π sections. For a Π network, if $Z_2 = Z_{2k}/m$, Eq. 10-8 gives

$$\frac{1}{Z_1} = \frac{1}{mZ_{1k}} + \frac{1}{Z_{2k}}\frac{1 - m^2}{4m} \tag{10-32}$$

9. *m*-derived Sections. It will be found that the two new impedances, Z_1 and Z_2 of Fig. 10-10*b*, are physically realizable if m is a real number between 0 and 1. Consider the low-pass T-section network of Fig. 10-11*b*.



(a) Prototype (b) m-Derived

FIG. 10-10. *m*-derived T section.

The series impedance, Z_1, is inductive in the m-derived filter as it is in the prototype, but smaller (since m is between 0 and 1). The shunt impedance, Z_2, must consist of two parts (see Eq. 10-31), one part capacitive like Z_{2k} but with a larger impedance, and the other part inductive like Z_{1k}.

A similar statement can be made about a high-pass T section. For

Π-section networks, Eq. 10-32 shows that there is a corresponding relation among the admittances.

From the derivation, we know that the T section of Fig. 10-11*b* has the same characteristic impedance as its prototype, whatever the value of *m*. Figure 10-9*a* is applicable. Any such sections can therefore be connected in cascade. But is it desirable to do so? Does such a filter section have valuable characteristics?

The most obvious feature of the *m*-derived section is that its series branch will be resonant and will therefore have zero impedance at one frequency. No signal can reach the load at this particular frequency, and attenuation (assuming lossless reactances) will be infinite. This frequency is designated f_∞; its value is easily computed.

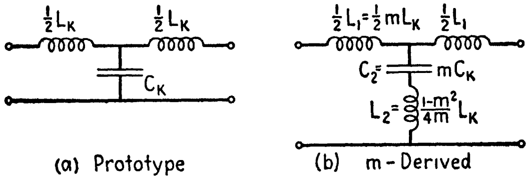


FIG. 10-11. *m*-derived low-pass T section.

The cutoff frequency is the frequency at which Z_0 becomes imaginary, and since Z_0 is the same for an *m*-derived section as for its prototype, it follows that *the cutoff frequency of an m-derived section is the same as the cutoff frequency of the prototype.* Hence, from Eq. 10-15, the cutoff frequency of an *m*-derived low-pass T section is

$$f_c = \frac{1}{\pi\sqrt{L_k C_k}} \tag{10-33}$$

The frequency of infinite attenuation is that which makes $X_2 = 0$:

$$\frac{1 - m^2}{4m}\omega_\infty L_k - \frac{1}{m\omega_\infty C_k} = 0 \tag{10-34}$$

which gives

$$\omega_\infty^2 = \frac{4}{(1 - m^2)L_k C_k} \tag{10-35}$$

Using Eq. 10-33,

$$f_\infty = \frac{f_c}{\sqrt{1 - m^2}} \tag{10-36}$$

Assuming that the cutoff frequency is fixed by the filter specifications, the most interesting deduction from this equation is that the frequency of infinite attenuation is dependent upon the arbitrary constant *m*. Hence, by appropriate choice of *m*, the designer can provide infinite attenuation at any desired frequency. This is the outstanding value of the *m*-derived filter section.

The amount of attenuation produced by an *m*-derived section can be computed from Eq. 10-25. Figure 10-12 shows typical attenuation curves for low-pass and high-pass *m*-derived sections of either T or Π configuration. It is interesting to compare these with Fig. 10-8 for the prototype section.

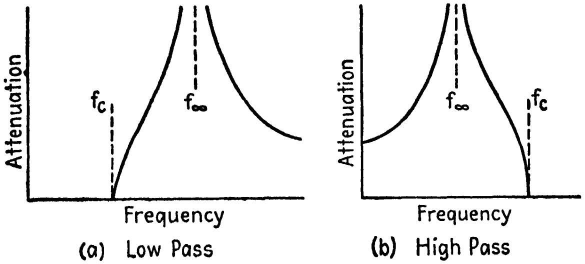


FIG. 10-12. Attenuation characteristics of *m*-derived filter sections.

Theoretically, the frequency of infinite attenuation can be placed as near the cutoff frequency as the designer wishes. If *m* approaches zero, f_∞ approaches f_c. Practically, the frequency of infinite attenuation should differ from the cutoff frequency by at least 2 to 5 per cent if ordinary high-*Q* coils are used.[4]

There are two reasons why sharper cutoff is not feasible. First, the size of the coils and condensers required becomes impracticable if *m* is very small. Consider that in the low-pass *m*-derived section of Fig. 10-11*b*, if *m* approached zero, L_1 would approach zero, C_2 would approach zero, and L_2 would approach infinity. The limit would be zero impedance in the series arms, and infinite impedance in the shunt. In the limit, as *m* approaches zero, the filter section is reduced to a mere link that serves only to connect the adjacent sections of the network and ceases to act as a filter. This is true of low and high pass, T and Π sections alike. Second, even if the impracticable circuit elements required for small *m* were built, they would fail to provide the desired high attenuation. All the foregoing discussion has neglected losses in the coils and condensers, and with most filters this leads to no serious discrepancies, but if *m* is small it turns out that losses are important and the filter fails to give the performance predicted on the basis of negligible loss. [Equations 10-3 and 10-4 (or 10-20) are not restricted to ideal filters, and may be used if it is necessary to compute attenuation in a T or Π section including the effect of loss.]

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[4] With quartz crystal filters, *Q* being higher, the cutoff may be sharper.

The possibility of connecting a number of different m-derived sections in cascade, all having the same cutoff frequencies and the same characteristic impedance, but with their various infinite-attenuation frequencies distributed throughout the stop band, is most attractive. The designer can provide high attenuation at all stop-band frequencies, with specially high attention at particular frequencies where it is most needed. This is the basis on which composite filters are usually designed. A composite filter can be made with as many sections as desired, but three sections are almost always enough, and two are often satisfactory.

10. Terminal Half Sections. One of the major difficulties of constant-k sections, lack of sharp cutoff, has now been remedied. The difficulty of matching that results from the variable characteristic impedance of the section remains to be overcome, for an m-derived section has a characteristic impedance as bad as its prototype.

An obvious solution of the matching problem is to provide a matching network that has constant impedance, equal to the load, when viewed from the load side, and variable impedance, equal to that of the filter section, when viewed from the filter side.[5] We must consider the possibility of devising such a matching section.

In exploring the situation we recognize that such a matching section would be unsymmetrical, and would look like a prototype or m-derived filter section when viewed from the filter side. A good practical solution was found by using half of a filter section as a matching network.

Figure 10-13 shows a half section with terminals aa' and bb'. Looking from the left, this is half a T section. When viewed from the right, it is half a Π section. It could be connected to a T section on the left at aa' and to a Π section on the right at bb', and impedances would match at both pairs of terminals. The half section transforms from T-section characteristic impedance on one side to Π-section characteristic impedance on the other.

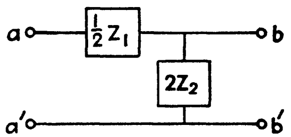

FIG. 10-13. A half section.

If the half section were half of a low-pass prototype section, as in Fig. 10-14a, it would serve to match the impedance of Fig. 10-9a to that of Fig. 10-9c. This impedance transformation is not suitable, however, for our purpose. We wish to match the impedance of a T section (or the impedance of a Π section) to the constant resistance of a load; the prototype half section transforms the impedance too greatly. We now have the following information: a prototype half section for which $m = 1$ provides too much of an impedance transformation, whereas a section for which $m = 0$ serves merely as a connecting link and provides no impedance transformation at all. This suggests that we investigate the impedance of an m-derived half section with m intermediate between 1 and 0.

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[5] Matching networks are discussed in Sec. 7, Chap. 9.

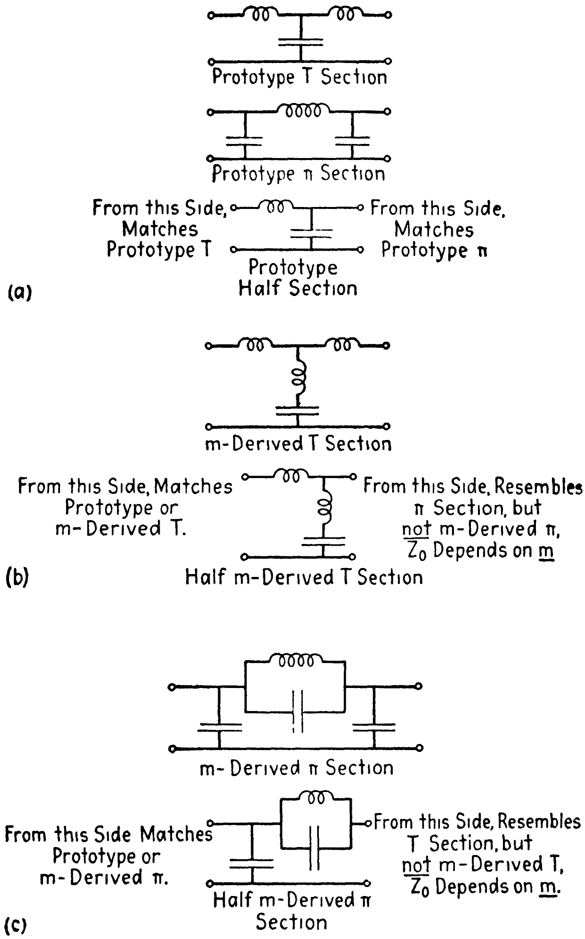

Prototype T Section
Prototype π Section
From this Side, Matches Prototype T
From this Side, Matches Prototype π
Prototype Half Section
(a)
m-Derived T Section
From this Side, Matches Prototype or m-Derived T.
From this Side, Resembles π Section, but not m-Derived π, Z0 Depends on m
(b)
Half m-Derived T Section
m-Derived π Section
From this Side Matches Prototype or m-Derived π.
From this Side, Resembles T Section, but not m-Derived T, Z0 Depends on m.
Half m-Derived π Section
(c)

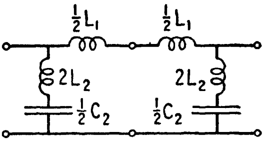
FIG. 10-14. Low-pass filter sections and half sections.

Half of an m-derived T section is shown in Fig. 10-14b. From the right, this looks like a Π section with

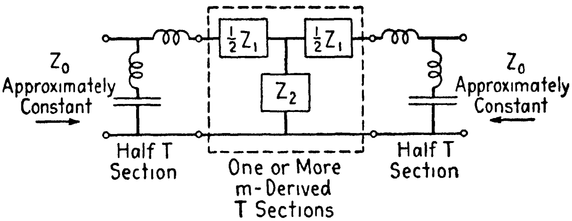
$$Z_1 = mZ_{1k} \tag{10-37}$$

$$Z_2 = \frac{Z_{2k}}{m} + Z_{1k}\frac{1 - m^2}{4m} \tag{10-38}$$

It must be emphasized that these are the proportions of an *m*-derived T section as in Fig. 10-14*b* and not of an *m*-derived Π section as in Fig. 10-14*c*. If two *m*-derived T half sections are put together as in Fig. 10-15*a*, they form a Π section, but it is *not* an *m*-derived Π and does not have the impedance indicated in Fig. 10-9*c*. Our next step is to find what impedance it does have.



(a) Two Half T Sections put Together as a π


Z_0 Approximately Constant
$\frac{1}{2}Z_1$
$\frac{1}{2}Z_1$
Z_2
Z_0 Approximately Constant
Half T Section
One or More m-Derived T Sections
Half T Section

(b) Two Half T Sections Matched to Intermediate T Sections

FIG. 10-15. Terminal half sections.

This is done by substituting Z_1 and Z_2 of Eqs. 10-37 and 10-38 into Eq. 10-8 for $Z_{0\pi}$. The result depends on the numerical value of m. The char-

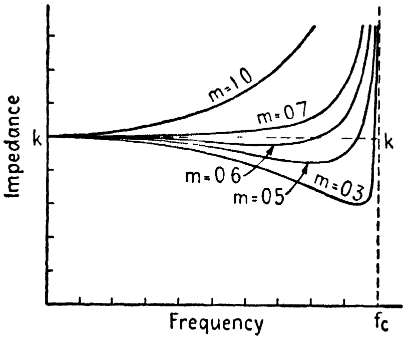


FIG. 10-16. Image (characteristic) impedance of the sections of Fig. 10-15. (Graph for low-pass half-T section; for high-pass, use reciprocal of frequency; for half-Π section, use reciprocal of impedance.)

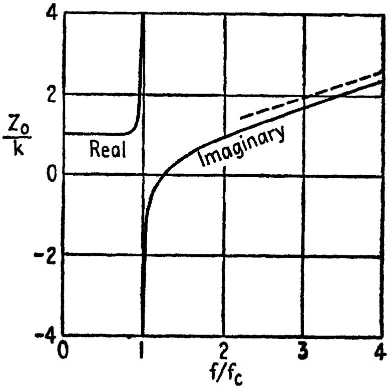


FIG. 10-17. Image (characteristic) impedance of terminal half-T sections of Fig. 10-15; low-pass *m*-derived filter with $m = 0.6$.

acteristic-impedance functions for various m's are plotted in Fig. 10-16. It is seen that if $m = 0.6$, our objective is very nearly achieved:[6] the characteristic impedance is nearly constant through most of the pass band. The value of the characteristic impedance of such a half section approximates $\sqrt{L_k/C_k}$ which, by Eq. 10-19, equals k.

The practical result is that a filter can be designed with one or more intermediate m-derived T sections, as in Fig. 10-15*b*, and with half an m-derived T section (for which $m = 0.6$) at each end. The intermediate filter sections will then be matched to each other and to the *internal* terminals of the end half sections, for all will have the impedance of Fig. 10-9*a* or *b*, while the end half sections will be matched *externally* to the *constant* impedance of input and output circuits.

The above statement applies to frequencies in the pass band. In the stop band, the characteristic impedance that is seen looking into the terminal half sections of the filter from the input or output circuits is purely reactive. The magnitude of it is highly variable; it can be computed, and is plotted in Fig. 10-17.

A filter that is terminated in half sections designed with $m = 0.6$ has a characteristic impedance Z_0 that varies less than 4 per cent through 86 per cent of the pass band. There is nothing in our approach to the problem to suggest that this is the best possible termination of a filter. As a matter

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[6] The following analysis indicates that $Z_{0\pi}$ for this kind of section is approximately constant if $m = 0.707$. This is the best approximation for frequencies far from cutoff, but $m = 0.6$ gives better average results throughout the pass band. If operation of the filter very close to cutoff is particularly important, use of $m = 0.5$ should be considered. See F. E. Terman, "Radio Engineers' Handbook," p. 233 (Ref. 35), for design curves.

$$Z_{0\pi}^2 = \left(\frac{Z_1 Z_2}{Z_{0T}}\right)^2 = \frac{k^4\left(1 + \frac{Z_{1k}}{Z_{2k}}\frac{1 - m^2}{4}\right)^2}{k^2\left(1 + \frac{Z_{1k}}{4Z_{2k}}\right)} \tag{10-39}$$

$$= k^2 \frac{1 + 2\frac{Z_{1k}}{Z_{2k}}\frac{1 - m^2}{4} + \cdots}{1 + \frac{Z_{1k}}{4Z_{2k}}} \qquad \left(\text{if } \frac{Z_{1k}}{Z_{2k}} \ll 1\right) \tag{10-40}$$

Placing

$$1 + 2\frac{Z_{1k}}{Z_{2k}}\frac{1 - m^2}{4} = 1 + \frac{Z_{1k}}{4Z_{2k}} \tag{10-41}$$

to make $Z_{0\pi}$ constant, we find

$$m^2 = \tfrac{1}{2} \qquad \text{and} \qquad Z_{0\pi} = k \tag{10-42}$$

of fact, other and more complicated terminal sections will give even better results. But for most practical purposes, the half-section termination is quite satisfactory.

11. Filter Design: Low-pass and High-pass. The essential facts about low-pass and high-pass T and Π filter sections have now been considered, but they have not been organized for design purposes. Ordinarily the designer starts with knowledge of the load resistance and the cutoff frequency. Steps in design are as follows.

1. Decide whether T or Π intermediate sections will be used. The choice is based on convenience.

2. Design the terminal half sections using formulas collected in Tables 10-1 or 10-2.

3. The terminal half sections, acting together as a filter section with $m = 0.6$, give infinite attenuation at 1.25 f_c for low-pass and 0.80 f_c for high-pass. Consider what other frequencies should have infinite attenuation to give adequate attenuation throughout the stop band, remembering that an additional filter section is required for each. Usually a prototype section ($m = 1$) is included to give high attenuation at stop-band frequencies far from cutoff. This one intermediate prototype section with the two terminal half sections may constitute an adequate filter if requirements are not too exacting. If quite sharp cutoff is required, an intermediate section with low m should be included. A filter with one intermediate m-derived section and one prototype section, together with the terminal half sections, is sufficient for almost any purpose.

4. Design the intermediate sections using the formulas of Table 10-1 or 10-2.

5. Arrange the sections in cascade and where two condensers or two coils appear in series or parallel, combine into a single condenser or coil.

An example of design is given in Sec. 13.

The formulas of Tables 10-1 and 10-2 are collected from the earlier sections of this chapter,[7] with transformations that put them in a form suitable for design. Thus L_k and C_k, inductance and capacitance of the prototype low-pass T section, are found from Eqs. 10-15 and 10-25 as follows:

$$f_c = \frac{1}{\pi\sqrt{L_kC_k}} \tag{10-43}$$

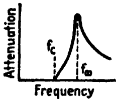
$$L_kC_k = \frac{1}{\pi^2 f_c^2} \tag{10-44}$$

---

[7] The arrangement is patterned after that used by F. E. Terman in various publications, including Refs. 8 and 35.

TABLE 10-1
DESIGN OF LOW-PASS FILTERS

Design Constants

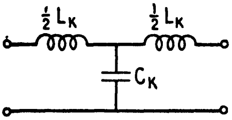


$R =$ load resistance

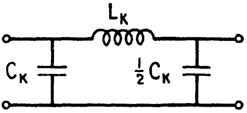
$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2}$$

Prototype Sections

$$(f_\infty = \infty, \qquad m = 1)$$



T Section

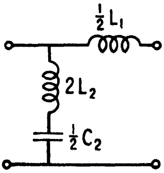


Π Section

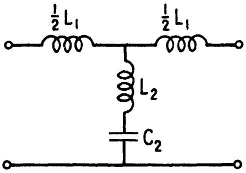
$$L_k = \frac{R}{\pi f_c}$$

$$C_k = \frac{1}{\pi f_c R}$$

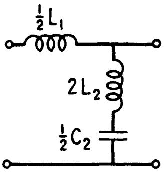
m-Derived T *Sections*



Terminal
Half Section
$m = 0.6$



Intermediate
Sections



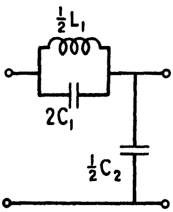
Terminal
Half Section
$m = 0.6$

$$L_1 = mL_k$$

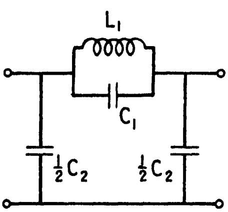
$$L_2 = \frac{1 - m^2}{4m} L_k$$

$$C_2 = mC_k$$

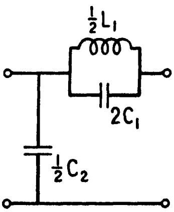
m-Derived Π *Sections*



Terminal
Half Section
$m = 0.6$



Intermediate
Sections



Terminal
Half Section
$m = 0.6$

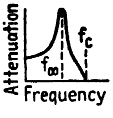
$$L_1 = mL_k$$

$$C_1 = \frac{1 - m^2}{4m} C_k$$

$$C_2 = mC_k$$

TABLE 10-2
DESIGN OF HIGH-PASS FILTERS

Design Constants

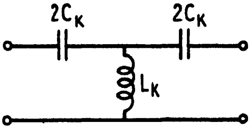


R = load resistance

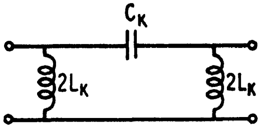
$$m = \sqrt{1 - \left(\frac{f_\infty}{f_c}\right)^2}$$

Prototype Sections

$(f_\infty = 0, \quad m = 1)$



T Section



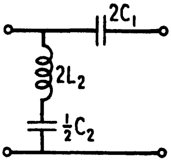
Π Section

$$L_k = \frac{R}{4\pi f_c}$$

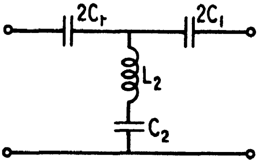
$$C_k = \frac{1}{4\pi f_c R}$$

---

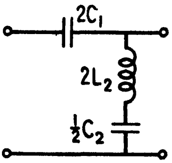
m-Derived T *Sections*



Terminal
Half Section
$m = 0.6$



Intermediate
Sections



Terminal
Half Section
$m = 0.6$

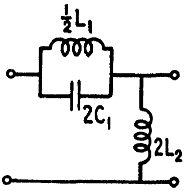
$$C_1 = \frac{C_k}{m}$$

$$C_2 = \frac{4m}{1 - m^2} C_k$$

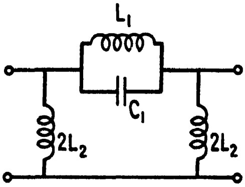
$$L_2 = \frac{L_k}{m}$$

---

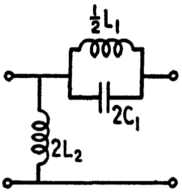
m-Derived Π *Sections*



Terminal
Half Section
$m = 0.6$



Intermediate
Sections



Terminal
Half Section
$m = 0.6$

$$C_1 = \frac{C_k}{m}$$

$$L_1 = \frac{4m}{1 - m^2} L_k$$

$$L_2 = \frac{L_k}{m}$$

By Eq. 10-28,

$$Z_{0T} = \sqrt{-X_1X_2\left(1 + \frac{X_1}{4X_2}\right)}$$

$$= \sqrt{\frac{L_k}{C_k}\left(1 - \frac{1}{4}\omega^2 L_k C_k\right)} \tag{10-45}$$

It is seen from Fig. 10-16 that the characteristic impedance should equal the load resistance R at zero frequency, regardless of the value of m. We therefore let $\omega = 0$ and set

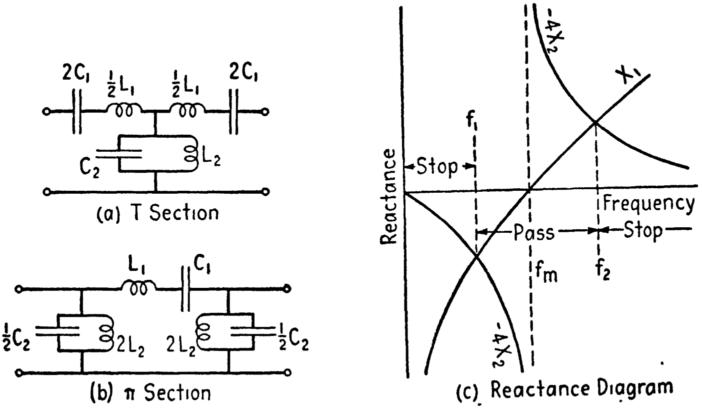
$$Z_{0T} = \sqrt{\frac{L_k}{C_k}} = R \tag{10-46}$$

Simultaneous solution of Eqs. 10-44 and 10-46 gives, for a low-pass T section,

$$L_k = \frac{R}{\pi f_c} \quad \text{and} \quad C_k = \frac{1}{\pi f_c R} \tag{10-47}$$

Prototype L and C for a high-pass T section, and for both kinds of Π sections, are found in a similar way.

The factor m is found by a rearrangement of Eq. 10-36. L_1, L_2, and C_2 then come from Eqs. 10-30 and 31. In the preceding pages, only the low-pass T section has been worked out in detail, but expressions for the other types are derived similarly.



(a) T Section

(b) π Section

(c) Reactance Diagram

FIG. 10-18. Band-pass inverse filter.

12. Band-pass Filters. It is often necessary to pass a band of frequencies between an upper cutoff frequency and a lower cutoff frequency. In carrier-current telephone transmission, for instance, a particular channel is separated from other channels by such a band-pass filter.

The two cutoff frequencies of a band-pass filter will be called f_1 and f_2, respectively. Frequencies below f_1 and frequencies above f_2 must be attenuated.

The principles of operation of a band-pass filter are the same as those of the high- and low-pass filters that have been considered. The prototype band-pass T section is shown in Fig. 10-18*a*, and the prototype Π section in *b*.

To see that these are indeed band-pass networks, the reactances are plotted in Fig. 10-18*c*. X_1 has the familiar form of the reactance of a series-resonant circuit, and $-4X_2$ is like the familiar parallel-resonance reactance curve, but inverted.

Note that X_1 and X_2 are both resonant at a frequency in the pass band, and that they both change sign at this frequency. They are resonant at the same frequency, for in order that the section may be a constant-*k* or inverse network, it is necessary that $L_1C_1 = L_2C_2$. It can be shown that the frequency at which the branches of the filter are resonant, called f_m the midfrequency, is such that $f_m^2 = f_1f_2$.

If X_1 did not change sign at the same frequency as X_2, the filter would have two pass bands separated by a stop band, in addition to the stop bands at high and low frequencies. However, our derivation requires the prototype section to be a constant-*k* or inverse network.

Like the prototype low- and high-pass filters, the prototype constant-*k* band-pass sections have variable image (characteristic) impedance, and lack sharp cutoff. Attenuation as a function of frequency is shown in Fig. 10-19; attenuation is infinite at very high and very low frequencies, but rises slowly from the cutoff frequencies.

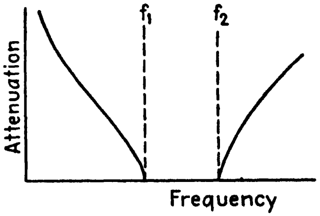


FIG. 10-19. Attenuation of constant-*k* band-pass sections.

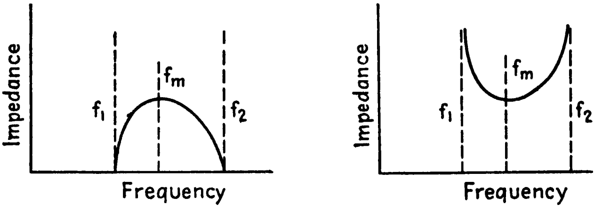
Characteristic impedance in the pass band is shown in Fig. 10-20. The T and Π sections have the same characteristic impedance at the midfrequency only, at which $Z_0 = \sqrt{L_2/C_1} = \sqrt{L_1/C_2} = k$. In the design of a band-pass filter, it is this midfrequency that is matched to the load resistance R; in this way it is analogous to the zero-frequency impedance of a low-pass filter (see Eqs. 10-19 and 10-46) and the infinite-frequency impedance of a high-pass filter.

To provide more desirable attenuation characteristics for a band-pass filter, *m*-derived sections are used. These have a frequency of infinite attenuation in each stop band, $f_{1\infty}$ and $f_{2\infty}$, as indicated in the figure of Table 10-3. There is an opportunity to select two arbitrary design constants, m_1 and m_2, in a band-pass section; the two frequencies of infinite attenuation can then be independently adjusted to any desired points in their respective stop bands.

If the two values of m are taken to be equal (if $m_1 = m_2 = m$), it is possible to select only one frequency of infinite attenuation, and the other is then specified by the relation

$$f_{1\infty}f_{2\infty} = f_1f_2 = f_m^2 \tag{10-48}$$

The two frequencies of infinite attenuation then appear in corresponding locations on opposite sides of the pass band. A small value of m provides sharp cutoff on both sides of the pass band. When m approaches its maximum value of 1, the points of infinite attenuation are near zero and infinite frequency, and the network is similar to the prototype section.



(a) T section (b) Π section

FIG. 10-20. Image (characteristic) impedance in the pass band of prototype and m-derived band-pass sections.

Equations 10-31 and 10-32 give the relations that must exist among the elements of an m-derived section, and apply to band-pass filter sections as well as to low-pass and high-pass sections. (This assumes a single value of m.) In an m-derived T section, for example, the shunt arm must contain an impedance similar to the shunt impedance of the prototype, but larger, together with an impedance similar to the series impedance of the prototype. The results are arranged in Table 10-3.

Band-pass sections of the m-derived type have the unsatisfactory characteristic impedance qualities shown in Fig. 10-20. As with low-pass and high-pass filters, terminal half sections are used to match the impedance of a band-pass filter to input and output circuits. As with low-pass and high-pass filters, the best over-all results are obtained by letting $m = 0.6$ for the terminal half sections, although this value may be varied a little for exceptional designs.

Design data providing for the independent selection of infinite-attenuation frequencies are given in Table 10-4.[8] Type I of this table is the most

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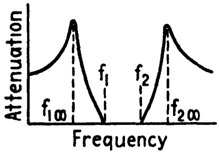
[8] Reproduced, by permission, from F. E. Terman's "Radio Engineers' Handbook" (Ref. 35). Much of the information is from Otto J. Zobel's papers in the *Bell System Technical Journal.*

TABLE 10-3
DESIGN OF BAND-PASS FILTERS

Design Constants



R = load resistance

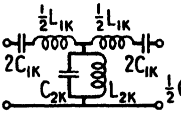
$$m = \sqrt{1 - \left[\frac{f_\infty(f_2 - f_1)}{f_\infty^2 - f_1 f_2}\right]^2}$$

$$(f_\infty = f_{1\infty} \quad \text{or} \quad f_{2\infty})$$

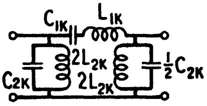
$$f_{1\infty} f_{2\infty} = f_1 f_2$$

Prototype Sections

$$(f_{1\infty} = 0, \qquad f_{2\infty} = \infty, \qquad m = 1)$$



T Section

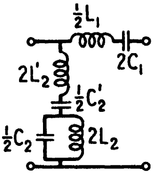


Π Section

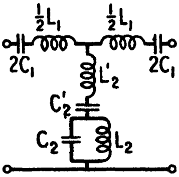
$$L_{1k} = \frac{R}{\pi(f_2 - f_1)} \qquad C_{1k} = \frac{f_2 - f_1}{4\pi f_1 f_2 R}$$

$$L_{2k} = \frac{R(f_2 - f_1)}{4\pi f_1 f_2} \qquad C_{2k} = \frac{1}{\pi(f_2 - f_1)R}$$

m-Derived T *Sections*



Terminal
Half Section
$m = 0.6$



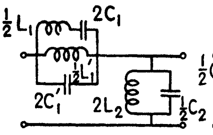
Intermediate
Sections

$$L_1 = mL_{1k} \qquad C_1 = \frac{C_{1k}}{m}$$

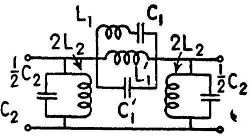
$$L_2 = \frac{L_{2k}}{m} \qquad C_2 = mC_{2k}$$

$$L_2' = \frac{1 - m^2}{4m} L_{1k} \qquad C_2' = \frac{4m}{1 - m^2} C_{1k}$$

m-Derived Π *Sections*



Terminal
Half Section
$m = 0.6$



Intermediate
Section

$$L_1 = mL_{1k} \qquad C_1 = \frac{C_{1k}}{m}$$

$$L_1' = \frac{4m}{1 - m^2} L_{2k} \qquad C_1' = \frac{1 - m^2}{4m} C_{2k}$$

$$L_2 = \frac{L_{2k}}{m} \qquad C_2 = mC_{2k}$$

TABLE 10-4. DESIGN OF BAND-PASS SECTIONS

Fundamental Relations

R = load resistance

f_1 = lower frequency limit of pass band

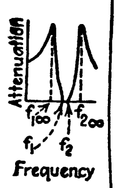
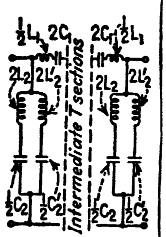
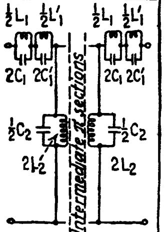
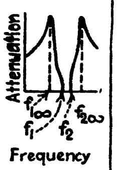
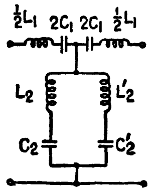
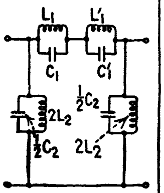
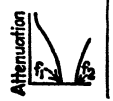
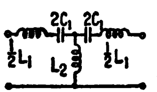
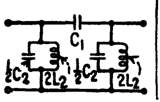
f_2 = higher frequency limit of pass band

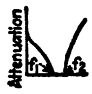
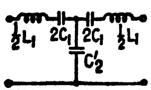
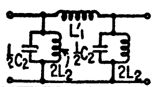
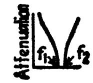
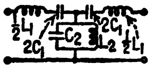
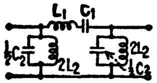
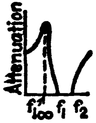
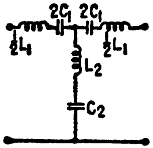
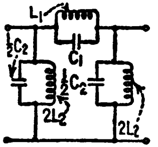
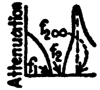
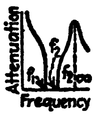
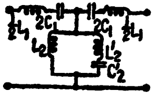
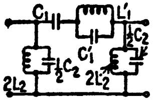
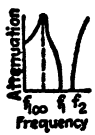
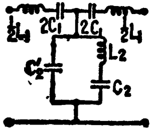
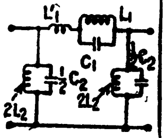
$f_{1\infty}$ = a frequency of very high attenuation in low-frequency attenuating band

$f_{2\infty}$ = a frequency of very high attenuation in high-frequency attenuating band

$$L_{1k} = \frac{R}{\pi(f_2 - f_1)} \qquad L_{2k} = \frac{(f_2 - f_1)R}{4\pi f_1 f_2} \qquad C_{1k} = \frac{f_2 - f_1}{4\pi f_1 f_2 R} \qquad C_{2k} = \frac{1}{\pi(f_2 - f_1)R}$$

Design of Sections

Type	Attenuation characteristic	A. Filters having T intermediate sections		B. Filters having π intermediate sections		Notation for both T and π sections
		Configuration	Formulas	Configuration	Formulas	
End ($m_1 = m_2$ = approximately 0.6)		 $\frac{1}{2}L_1$ $2C_1$ $2C_1'$ $\frac{1}{2}L_1$ $2L_2$ $2L_2'$ $2L_2$ $2L_2'$ $\frac{1}{2}C_2$ $\frac{1}{2}C_2'$ $\frac{1}{2}C_2$ $\frac{1}{2}C_2'$ Intermediate T sections	$L_1 = m_1L_{1k}$ $L_2 = aL_{1k}$ $L_2' = cL_{1k}$ $C_1 = \frac{C_{1k}}{m_2}$ $C_2 = \frac{C_{1k}}{b}$ $C_2' = \frac{C_{1k}}{d}$	 $\frac{1}{2}L_1$ $\frac{1}{2}L_1'$ $\frac{1}{2}L_1$ $\frac{1}{2}L_1'$ $2C_1$ $2C_1'$ $2C_1$ $2C_1'$ $\frac{1}{2}C_2$ $\frac{1}{2}C_2$ $2L_2'$ $2L_2$ Intermediate π sections	$L_1 = \frac{L_{2k}}{b}$ $L_1' = \frac{L_{2k}}{d}$ $L_2 = \frac{L_{2k}}{m_2}$ $C_1 = aC_{2k}$ $C_1' = cC_{2k}$ $C_2 = m_1C_{2k}$	$g = \sqrt{\left(1-\frac{f_{1\infty}^2}{f_1^2}\right)\left(1-\frac{f_{1\infty}^2}{f_2^2}\right)}$ $h = \sqrt{\left(1-\frac{f_1^2}{f_{2\infty}^2}\right)\left(1-\frac{f_2^2}{f_{2\infty}^2}\right)}$ $m_1 = \frac{\frac{f_1f_2}{f_{2\infty}^2}g + h}{1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}}$ $m_2 = \frac{g + \frac{f_{1\infty}^2}{f_1f_2}h}{1 - \frac{f_{1\infty}^2}{f_{2\infty}^2}}$, $a = \frac{(1-m_1^2)f_{2\infty}^2}{4gf_1f_2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right) = \frac{(1-m_2^2)f_1f_2}{4gf_{1\infty}^2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$ $b = \frac{(1-m_2^2)}{4g}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$ $c = \frac{(1-m_1^2)}{4h}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$ $d = \frac{(1-m_1^2)f_{2\infty}^2}{4hf_1f_2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right) = \frac{(1-m_2^2)f_1f_2}{4hf_{1\infty}^2}\left(1-\frac{f_{1\infty}^2}{f_{2\infty}^2}\right)$ These formulas apply for both end and Type I sections when ($m_1 = m_2$), $g = h, a = d, b = c, f_{1\infty} = \frac{f_1f_2}{f_{2\infty}}, m_1 = m_2 = \frac{h}{1 - \frac{f_1f_2}{f_{2\infty}^2}}$ and $f_{2\infty}^2 = \frac{f_1^2+f_2^2-2m^2f_1f_2}{2(1-m^2)} + \left[\left(\frac{f_1^2+f_2^2-2m^2f_1f_2}{2(1-m^2)}\right)^2 - f_1^2f_2^2\right]^{1/2}$
I		 $\frac{1}{2}L_1$ $2C_1$ $2C_1$ $\frac{1}{2}L_1$ L_2 L_2' C_2 C_2'	$L_1 = m_1L_{1k}$ $L_2 = aL_{1k}$ $L_2' = cL_{1k}$ $C_1 = \frac{C_{1k}}{m_2}$ $C_2 = \frac{C_{1k}}{b}$ $C_2' = \frac{C_{1k}}{d}$	 L_1 L_1' C_1 C_1' $2L_2$ $\frac{1}{2}C_2$ $\frac{1}{2}C_2$ $2L_2'$	$L_1 = \frac{L_{2k}}{b}$ $L_1' = \frac{L_{2k}}{d}$ $L_2 = \frac{L_{2k}}{m_2}$ $C_1 = aC_{2k}$ $C_1' = cC_{2k}$ $C_2 = m_1C_{2k}$	
II $f_{1\infty} = 0$ $f_{2\infty} = f_2$		 $2C_1$ $2C_1$ $\frac{1}{2}L_1$ L_2 $\frac{1}{2}L_1$	$L_1 = \frac{f_1R}{\pi f_2(f_2 - f_1)}$ $L_2 = \frac{(f_1 + f_2)R}{4\pi f_1f_2}$ $C_1 = C_{1k}$	 C_1 $\frac{1}{2}C_2$ $2L_2$ $\frac{1}{2}C_2$ $2L_2$	$C_1 = \frac{f_1 + f_2}{4\pi f_1f_2R}$ $C_2 = \frac{f_1}{\pi f_2(f_2 - f_1)R}$ $L_2 = L_{2k}$	

III $f_{1\infty} = f_1$ $f_{2\infty} = \infty$		 $\frac{1}{2}L_1$ $2C_1$ $2C_1$ $\frac{1}{2}L_1$ C_2'	$L_1 = L_{1k}$ $C_2' = \frac{1}{\pi(f_1 + f_2)R}$ $C_1 = \frac{f_2 - f_1}{4\pi f_1^2 R}$	 L_1' $\frac{1}{2}C_2$ $\frac{1}{2}C_2$ $2L_2$ $2L_2$	$L_1' = \frac{R}{\pi(f_1 + f_2)}$ $L_2 = \frac{(f_2 - f_1)R}{4\pi f_1^2}$ $C_2 = C_{2k}$	
IV $f_{1\infty} = 0$ $f_{2\infty} = \infty$		 $\frac{1}{2}L_1$ $2C_1$ C_2 L_2 $2C_1$ $\frac{1}{2}L_1$	$L_1 = L_{1k}$ $L_2 = L_{2k}$ $C_1 = C_{1k}$ $C_2 = C_{2k}$	 L_1 C_1 $\frac{1}{2}C_2$ $2L_2$ $2L_2$ $\frac{1}{2}C_2$	$L_1 = L_{1k}$ $L_2 = L_{2k}$ $C_1 = C_{1k}$ $C_2 = C_{2k}$	
V $f_{2\infty} = f_2$		 $2C_1$ $2C_1$ $\frac{1}{2}L_1$ $\frac{1}{2}L_1$ L_2 C_2	$L_1 = m_1L_{1k}$ $L_2 = \frac{(1 - m_1^2)}{4m_1}L_{1k}$ $C_1 = \frac{C_{1k}}{m_1}$ $C_2 = \frac{4m_2}{1 - m_2^2}C_{1k}$ See notation for m_1 and m_2	 L_1 $\frac{1}{2}C_2$ C_1 $\frac{1}{2}C_2$ $2L_2$ $2L_2$	$L_1 = \frac{4m_2}{1 - m_2^2}L_{2k}$ $L_2 = \frac{L_{2k}}{m_2}$ $C_1 = \frac{(1 - m_1^2)}{4m_1}C_{2k}$ $C_2 = m_1C_{2k}$ See notation for m_1 and m_2	$m_1 = \frac{f_1}{f_2}m_2$, $m_2 = \sqrt{\frac{1 - \frac{f_{1\infty}^2}{f_1^2}}{1 - \frac{f_{1\infty}^2}{f_2^2}}}$
VI $f_{1\infty} = f_1$		Same circuit as above for Type V	Same formulas as above for Type V. See notation for m_1 and m_2	Same circuit as above for Type V	Same formulas as above for Type V See notation for m_1 and m_2	$m_1 = \sqrt{\frac{1 - \frac{f_2^2}{f_{2\infty}^2}}{1 - \frac{f_1^2}{f_{2\infty}^2}}}$ $m_2 = \frac{f_1}{f_2}m_1$
VII $f_{1\infty} = 0$		 $\frac{1}{2}L_1$ $2C_1$ L_2 $2C_1$ $\frac{1}{2}L_1$ L_2' C_2'	$L_1 = m_1L_{1k}$ $L_2 = aL_{1k}$ $L_2' = \frac{(1 - m_1^2)}{4h}L_{1k}$ $C_1 = C_{1k}$ $C_2' = \frac{h}{a}C_{1k}$	 C_1 L_1' $\frac{1}{2}C_2$ C_1' $\frac{1}{2}C_2$ $2L_2$ $2L_2$	$L_1' = \frac{h}{a}L_{2k}$ $L_2 = L_{2k}$ $C_1 = aC_{2k}$ $C_2 = m_1C_{2k}$ $C_1' = \frac{(1 - m_1^2)}{4h}C_{2k}$	$h = \sqrt{\left(1 - \frac{f_1^2}{f_{2\infty}^2}\right)\left(1 - \frac{f_2^2}{f_{2\infty}^2}\right)}$ $m_1 = \frac{f_1f_2}{f_{2\infty}^2} + h$ $a = \frac{(1 - m_1^2)f_{2\infty}^2}{4f_1f_2}$
VIII $f_{2\infty} = \infty$		 $\frac{1}{2}L_1$ $2C_1$ $2C_1$ $\frac{1}{2}L_1$ L_2 C_2' C_2	$L_1 = L_{1k}$ $L_2 = \frac{d}{g}L_{1k}$ $C_1 = \frac{C_{1k}}{m_2}$ $C_2 = \frac{4g}{1 - m_2^2}C_{1k}$ $C_2' = \frac{C_{1k}}{d}$	 L_1' L_1 C_1 $\frac{1}{2}C_2$ $\frac{1}{2}C_2$ $2L_2$ $2L_2$	$L_1 = \frac{4g}{1 - m_2^2}L_{2k}$ $L_2 = \frac{L_{2k}}{m_2}$ $L_1' = \frac{L_{2k}}{d}$ $C_1 = \frac{d}{g}C_{2k}$ $C_2 = C_{2k}$	$g = \sqrt{\left(1 - \frac{f_{1\infty}^2}{f_1^2}\right)\left(1 - \frac{f_{1\infty}^2}{f_2^2}\right)}$ $m_2 = g + \frac{f_{1\infty}^2}{f_1f_2}$ $d = \frac{(1 - m_2^2)f_1f_2}{4f_{1\infty}^2}$

general type; although the network does not appear identical with that of Table 10-3, it is electrically equivalent. The other types are various degenerate forms of Type I in which coils or condensers reduce to zero. Type IV will be recognized as the prototype.

The design of a band-pass filter must begin with a knowledge of the upper and lower cutoff frequencies and the load resistance. Steps in the design are then exactly as outlined in Sec. 11. Tables 10-3 or 10-4 give the essential formulas.

When several frequency channels are to be separated by band-pass filters, several filters must be connected in parallel (or in series). This cannot be done without care, for it is apparent that each filter will affect the operation of the others. Each filter will act the same, at any frequency, as would an impedance equal to the input impedance to that filter. In the pass band, the input impedance to a filter is R. In the stop band, the input impedance is almost purely reactive and is approximately equal to the image (characteristic) impedance of the input section of the filter. The curves of Fig. 10-17 (although this figure is specifically for a low-pass filter) give a general idea of the manner in which image impedance can be expected to vary in the stop band. Schemes that permit connecting filters in parallel or series will be found in detailed treatises on filters [see Terman's Handbook (Ref. 35) and references therein].

13. Design Example. Design a high-pass filter with cutoff at 1,000 cycles per second, to match 700-ohm input and output circuits. It is required that attenuation be at least 20 decibels (power ratio 100 to 1) at all frequencies from 0 to 900 cycles. Follow the steps of design procedure enumerated in Sec. 11.

1. Π-type intermediate sections will be used. There is no reason for this choice; it is purely arbitrary in this instance.

2. Using Table 10-2,

$$L_k = \frac{700}{4\pi \cdot 1{,}000} = 0.0557 \text{ henry}$$

$$C_k = \frac{1}{4\pi \cdot 1{,}000 \cdot 700} = 0.1136 \times 10^{-6} \text{ farad}$$

For the terminal half sections, let $m = 0.6$. Then

$$\frac{4m}{1 - m^2} = 3.75$$

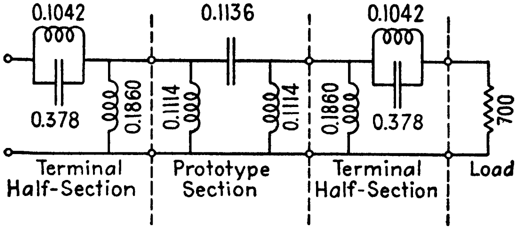
$$C_1 = \frac{0.1136}{0.6} 10^{-6} = 0.1890 \times 10^{-6}$$

$$L_1 = 3.75 \times 0.0557 = 0.2085$$

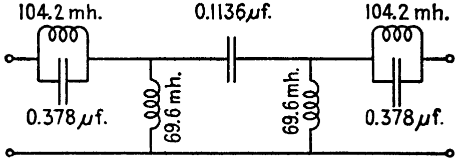
$$L_2 = \frac{0.0557}{0.6} = 0.0930$$

These numerical values for the terminal half sections are now entered in the diagram of Fig. 10-21*a*.

3. What intermediate sections are needed? For adequate attenuation at low frequency an intermediate prototype section ($m = 1$) will be used. The specified sharpness of cutoff is not great, and it may be unnecessary to use a second intermediate section. (If sharp cutoff were required, an intermediate section with $f_\infty = 900$ or 950 cycles would be added.) Let us assume that the only intermediate section will be the prototype section and compute the attenuation to see if it is satisfactory.



(*a*) Sections as Designed



(b) Elements Merged

FIG. 10-21. Example of high-pass filter design.

4. Elements of the prototype section have already been computed in step 2, above, and may be entered in the diagram of Fig. 10-21*a*.

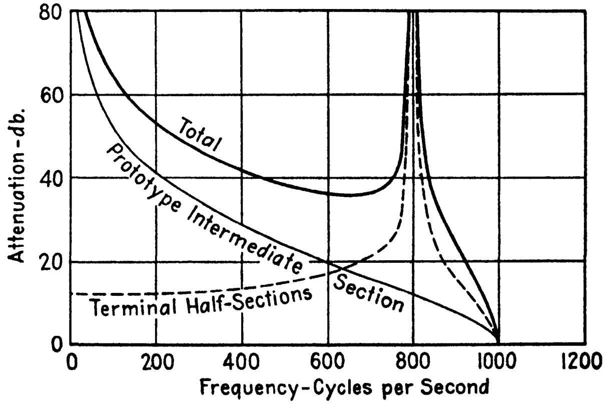


FIG. 10-22. Attenuation of individual sections and of composite filter of Fig. 10-21.

5. Shunt inductances in the filter sections are found to be connected in parallel. These can be combined into single coils, as shown in Fig. 10-21*b*.

Attenuation will now be computed. Equation 10-23 is used. Let us first compute attenuation in the two terminal half sections. Letting α represent the attenuation per complete section,

$$\cosh \alpha = \left| 1 + \frac{1}{2}\frac{Z_1}{Z_2} \right| = \left| 1 - \frac{1}{2}\frac{L_1}{\omega C_1 L_2\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)} \right|$$

Results, computed as a function of ω, are shown in the dash curve of Fig. 10-22.

Attenuation provided by the intermediate prototype section is next computed. The same formula is used:

$$\cosh \alpha = \left| 1 + \frac{1}{2}\frac{Z_1}{Z_2} \right| = \left| 1 - \frac{1}{2\omega^2 L_k C_k} \right|$$

Results are plotted in Fig. 10-22.

Total attenuation in the filter made up of the prototype section and the two half sections in cascade is the sum of the individual attenuations, and this is also shown in Fig. 10-22. It was specified in the statement of the problem that attenuation was to be at least 20 decibels at frequencies below 900 cycles. This much attenuation is provided; thus the simple filter of Fig. 10-21 is adequate. If the specifications were more rigorous, another intermediate section would be used.

14. Nonideal Behavior of Practical Filters. Imperfect Termination. Filter computations, as in the above example, include two idealizations. The equation for computing attenuation is based on the assumption that the filter is terminated in its image (characteristic) impedance. As we have seen (Fig. 10-17), this assumption is reasonably valid in most of the pass band, for the filter was designed to make it so. As the cutoff frequency is approached, however, the assumption becomes less exact, and throughout the stop band it is totally false.

In the pass band, near cutoff, the effect of the unavoidable variation of Z_0 is to introduce a small amount of undesired attenuation. Sharpness of cutoff is somewhat lessened.

At first thought, it would seem that computations with stop-band frequencies would be entirely invalidated by the fact that the actual load has neither the angle nor the magnitude of the characteristic impedance. However, we may consider that in the stop band the attenuation of the filter is high; therefore very little of the applied signal reaches the load; therefore it is not greatly significant what the nature of the load may be, for it cannot have much effect. This argument is slightly specious, but the conclusion is none the less true.

Evidence is furnished by the computation given at the beginning of the chapter. In Sec. 2, the performance of a network with T configuration was considered. This network may now be recognized as being composed of two half sections of an *m*-derived Π, rearranged to form a T. It is the same T that would be obtained if the prototype section of Fig. 10-21*a* were omitted and the two end half sections were directly connected. Indeed, the numerical values of inductance and capacitance given in Sec. 2 are the same as the results of the design problem of Sec. 13. Hence a comparison is available between the results obtained in Sec. 2 by the laborious but exact methods of elementary computation, and the results of Sec. 13 in which we applied to the same network the idealized method that assumes the filter to be terminated in its characteristic impedance.

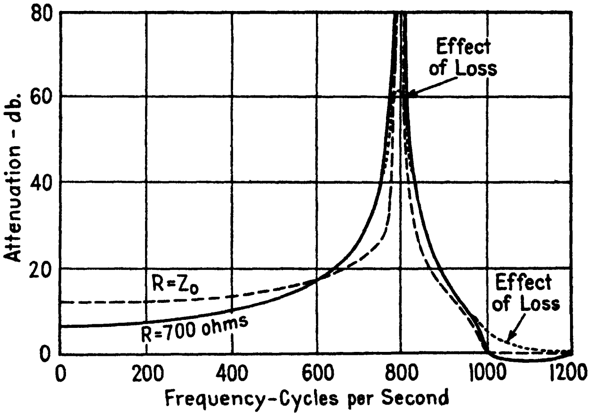


FIG. 10-23. Comparison of idealized and exact computation of attenuation.

Attenuation curves from the two computations are compared in Fig. 10-23. These are replotted from Figs. 10-3*b* and 10-22. Agreement through the pass band is quite good. It is interesting to notice that there is a small "negative attenuation" near cutoff; this means that the load actually receives more power at these frequencies with the filter present than with it absent, owing to a resonant rise of voltage in the filter network.[9] (A purely reactive filter is assumed.)

In the stop band, the two curves agree in general shape, but not in detail. The difference, however, is only a few decibels, and is of no practical significance in parts of the stop band with high attenuation. When a section of this type is used as part of the composite filter of Fig. 10-22, an uncertainty of a few decibels in the *total* attenuation is not important. Since all impedance matches between sections of an m-derived filter are perfect, with a poor match (for stop-band frequencies) only at the load, the more the total attenuation, the less the importance of the imperfect match at the load.

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[9] Strictly speaking, the curve from Fig. 10-3*b* is more precisely an illustration of *insertion loss* than attenuation. "The insertion loss at a given frequency caused by the insertion of apparatus in a transmission system is the ratio, expressed in decibels, of the powers at that frequency delivered to that part of the system beyond the point of insertion before and after the insertion." ("American Standard Definitions of Electrical Terms," Ref. 1.) Insertion loss includes the effect of mismatch as well as attenuation.

The general conclusion is that attenuation may be computed by Eq. 10-23 with reasonable accuracy, although a certain approximation is involved.

Coil Resistance and Losses. The other idealization, which appears in all the work of this chapter, is the neglect of power loss in the filter network. Purely reactive elements have been assumed. The actual situation is that practical condensers may have negligible losses, but this is never true of coils.

The effect of loss is to prevent filter operation from being as extreme as is predicted:

1. In the pass band, attenuation is low but not zero.
2. At resonant frequencies, attenuation is high but not infinite.
3. Cutoff is not as sharp as computed.

A dotted line in Fig. 10-23 illustrates these three effects of loss. At pass-band frequencies beyond the range of the diagram, the attenuation owing to loss is practically negligible.

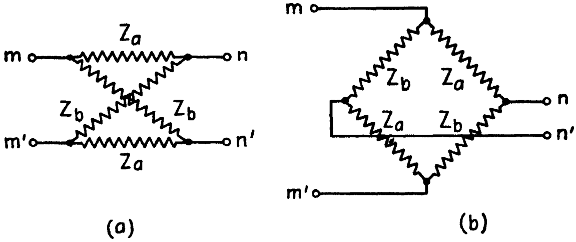
This dotted curve assumes the use of inductance coils of good quality. The effect of loss is secondary if the coils have a Q of 15 or more. Coil design naturally depends on the range of frequency of operation. Values of Q as high as 400 can be obtained with coils having molybdenum-permalloy dust cores; these are used for telephone filters at frequencies up to about 30 kilocycles. At higher frequencies, the piezoelectric resonance of quartz crystals is used to provide resonant elements with extremely high Q (several thousand) for filter networks. At ultra-high frequencies, open or short-circuited lengths of transmission line can be used to provide resonant elements for filters, and Q of several thousand is easily obtained.

15. Lattice Filters. The composite ladder filter sections that have been discussed are satisfactory for many practical applications. We have by no means explored their ultimate possibilities, but the m-derived sections, although a restricted classification even among ladder filters, are the most practical type for most uses.

For complete generality, the lattice-network filter should be studied. A lattice network is shown in Fig. 10-24. It can be shown that the impedance properties of a lattice network can be as general as the impedance properties of any symmetrical four-terminal network. A lattice will do anything that T or Π sections will do, and may be designed with other characteristics not obtainable with a ladder network.

The conventional arrangement of elements in a lattice network is shown in Fig. 10-24*a*, and in Fig. 10-24*b* the same elements are rearranged to show that the lattice is a bridge circuit. It is evident at once that if $Z_a = Z_b$, a signal may be applied to one pair of terminals and there will be no output at the other pair of terminals.

The impedance and transmission characteristics of a lattice can be studied in the same manner as those of Π and T sections. The starting point is to write the impedance at one pair of terminals, the other pair being first open-circuited and then short-circuited. By reference to the figure, these open- and short-circuit impedances are, respectively,



(a) (b)

FIG. 10-24. A lattice network.

$$Z_{op} = \tfrac{1}{2}(Z_a + Z_b) \tag{10-49}$$

$$Z_{sh} = 2\frac{Z_aZ_b}{Z_a + Z_b} \tag{10-50}$$

Referring again to Eq. 7-75 for image (characteristic) impedance,

$$Z_0^2 = Z_{op}Z_{sh} \tag{10-51}$$

Transmission is determined from Eq. 7-79:

$$\cos(\beta - j\alpha) = \frac{1}{\sqrt{1 - Z_{sh}/Z_{op}}} \tag{10-52}$$

or, alternatively, from Eq. 7-80,

$$\cosh(\alpha + j\beta) = \frac{1}{\sqrt{1 - Z_{sh}/Z_{op}}} \tag{10-53}$$

Using the impedance values of Eqs. 10-49 and 10-50 in Eq. 10-51, there results:

$$Z_0 = \sqrt{Z_aZ_b} \tag{10-54}$$

The same impedance values in Eq. 10-52 give

$$\cos(\beta - j\alpha) = \frac{1}{\sqrt{1 - \dfrac{4Z_bZ_a}{(Z_b + Z_a)^2}}} = \frac{Z_b + Z_a}{Z_b - Z_a} \tag{10-55}$$

A trigonometric change gives

$$\tan\left(\frac{\beta - j\alpha}{2}\right) = \sqrt{\frac{1 - \cos(\beta - j\alpha)}{1 + \cos(\beta - j\alpha)}} = \sqrt{-\frac{Z_a}{Z_b}} \tag{10-56}$$

Finding α and β from Eq. 10-56 requires evaluating the inverse tangent function. If the right-hand member of the equation is real, the argument of the tangent is real, and β exists but α is zero; this is true if Z_a and Z_b are of opposite sign. If the right-hand member is imaginary, α will exist, and β will be either 0 or π. It is therefore seen that the lattice, properly terminated, will pass frequencies for which Z_a and Z_b have opposite sign, and will attenuate frequencies for which Z_a and Z_b have the same sign.

More detailed examination of the complex tangent function leads to the following conclusions:[10]

Transmission band or pass band:

Z_a/Z_b is negative (and real),

$$\alpha = 0$$
$$\beta = \pm 2\tan^{-1}\sqrt{-\frac{Z_a}{Z_b}} \tag{10-57}$$

(The sign of β is the same as the sign of X_a.)

Attenuation band or stop band:

Z_a/Z_b is positive (real) and less than 1,

$$\alpha = 2\tanh^{-1}\sqrt{\frac{Z_a}{Z_b}}$$
$$\beta = 0 \tag{10-58}$$

Z_a/Z_b is positive (real) and greater than 1,

$$\alpha = 2\tanh^{-1}\sqrt{\frac{Z_b}{Z_a}}$$
$$\beta = \pi \tag{10-59}$$

---

[10] These conclusions may be reached from a study of a chart of the tangent function as given by Jahnke and Emde (Ref. 11), or by expanding the tangent function by Formula 25*a* of Table II.

The image impedance (characteristic impedance) of any lattice is $\sqrt{Z_aZ_b}$; the input impedance to any lattice that is terminated in impedance equal to $\sqrt{Z_aZ_b}$ is also $\sqrt{Z_aZ_b}$. (This fact can be verified by elementary circuit methods if desired.)

The practical problem in designing a lattice, as with any other filter, is to have the image impedance Z_0 as nearly constant as possible throughout the pass band, and to have the attenuation as high as possible in the stop band. The ideal lattice would have $Z_a = 1/Z_b$ in the transmission or pass band, for this would make Z_0 constant, and $Z_a = Z_b$ in the attenuation or stop band, for this would make attenuation infinite.

Clearly, it is too much to expect that the relation between Z_a and Z_b will change abruptly from one to the other of these radically different conditions. Nevertheless, this ideal can be approached quite closely by making Z_a and Z_b both fairly complicated networks of reactive elements. The art of design is to come as near to doing so as is economically justifiable.

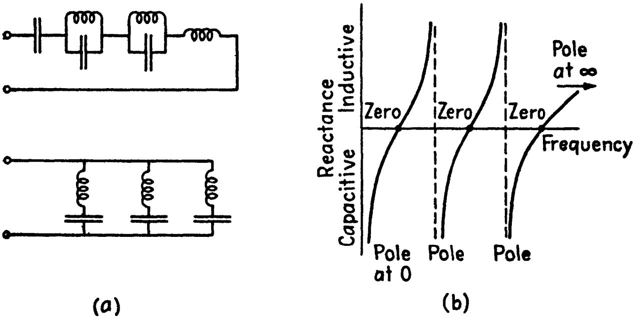


FIG. 10-25. Equivalent reactive networks, and their reactance characteristic.

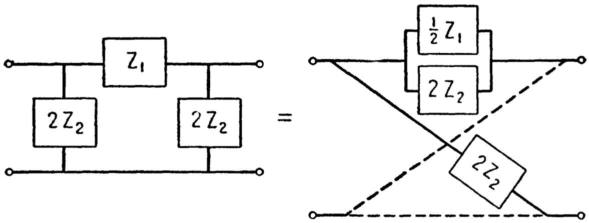
In designing a lattice filter, it is simple to control Z_0 and α independently, by choice of Z_a and Z_b (see Eqs. 10-54 and 10-56).

It is evident that infinite attenuation can be achieved at all frequencies by making Z_a and Z_b identical elements. The result (see Fig. 10-24*b*) is a balanced bridge. Such an arrangement does not seem to have any practical value as a filter.

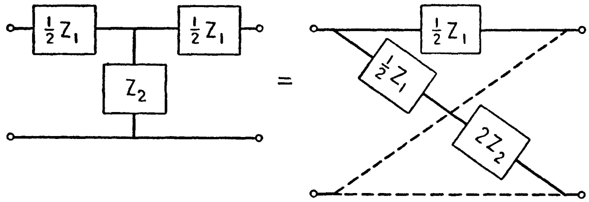
By making Z_a and Z_b inverse impedances (a coil in one arm and a condenser in the other being the simplest possible example) a network is obtained that has no attenuation at any frequency if it is properly terminated. It is called an *all pass* section. It produces a phase shift β that is dependent on the frequency as in Eq. 10-57. It is useful for phase correction, as, for example, to cancel out delay distortion at the end of a transmission line.

The methods of design of lattice filters have little to do with transmission-line theory. Moreover, many communication engineers never need them. Therefore they will not be given here, but the following brief outline will suggest the general principles used.

If reactance is plotted as a function of frequency, there are points at which reactance is zero, called *zeros*, and points at which it is infinite, called *poles* (see Fig. 10-25). The sign of the reactance changes at each pole and at each zero. For any network, poles and zeros alternate. Within the pass band of a lattice filter, if Z_a has a pole, Z_b must have a zero, and vice versa. In the stop band, if Z_a has a pole, Z_b must have a pole, and if Z_a has a zero, so must Z_b. If either Z_a or Z_b has a pole or zero, and the other impedance does not have either, the frequency at which it occurs is a cutoff frequency, for here the impedance of one arm of the filter changes sign and that of the other arm does not.



(a) Pi Network Transformed to Lattice



(b) T Network Transformed to Lattice

FIG. 10-26. Equivalence of lattice and ladder sections. (It is understood that the lattices are symmetrical; the dash lines indicate arms identical with those shown.)

Foster's reactance theorem gives the reactance of a two-terminal network in terms of the frequencies at which there are poles and zeros. With the use of this theorem, design becomes a problem of locating poles and zeros in a lattice in such a way that the ideal filter conditions are approached as nearly as possible. There are various practical schemes for doing this (following Cauer and Bode; Refs. 10 and 35) which will not be given here.

Any Π or T network can be transformed to an equivalent lattice. The equivalent networks are shown in Fig. 10-26. It does not follow that any lattice can be transformed to a Π or T. If a lattice can be so transformed, however, there may be some practical advantage in building the filter as a Π or T even though it was designed as a lattice; usually fewer coils and condensers are needed.

16. Balance of Networks. Clearly, all lattice networks are balanced to neutral. Impedance in one line is identical with that in the other.

Π and T networks may be either balanced or unbalanced. Their names come from the unbalanced arrangements, in which all the series impedance is placed in one side. If half the series impedance is placed in each side of the circuit, the Π or T becomes balanced (see Fig. 10-27). The balanced Π is sometimes called a *square network*, and the balanced T an H *network*, for obvious reasons.

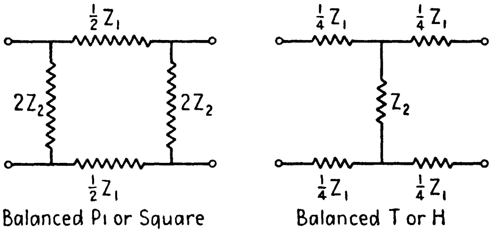


FIG. 10-27. Balanced Π and balanced T sections, equivalent to the unbalanced sections of Figs. 10-26 and 10-5.

A balanced Π or T requires more coils and condensers than does the equivalent unbalanced Π or T. Otherwise, the desirability of one form or the other is dictated by the input and output circuits. For some uses, the balance to ground provided by the square and H sections is required. For others, common connection of the shunt elements (as to ground) is advantageous, and this is possible only with the unbalanced Π or T. No general statement is possible.

17. Network Theory. In conclusion, it should be mentioned again that beautiful and extremely powerful methods of filter synthesis are provided by general network theory. The methods of this chapter are simple and relatively restricted. However, these simple methods provide for the design needs of the great majority of electrical engineers, and these concepts of filter operation that are based on transmission-line theory make use of the most expedient if not the most elegant approach to the subject.

PROBLEMS

10-1. Draw typical curves of reactance as a function of frequency for the following reactive elements: (*a*) simple inductance, (*b*) simple capacitance, (*c*) inductance and capacitance in series, and (*d*) inductance and capacitance in parallel. Indicate resonant frequencies.

Note: These typical curves will be used in the next several problems as they are in Figs. 10-6, 10-7, and 10-18.

10-2. Plotting X_1 and $-4X_2$, as in Fig. 10-6, indicate the pass and stop bands of a T or Π filter section in which X_1 is inductance and X_2 is inductance and capacitance in series. Classify as to high, low, or band pass, and mark the frequency or frequencies of infinite attenuation.

10-3. Repeat Prob. 10-2 for a section in which X_1 is L and C in parallel and X_2 is L.

10-4. Repeat Prob. 10-2 for a section in which X_1 is L and C in parallel and X_2 is C.

10-5. Repeat Prob. 10-2 for a section in which X_1 is L and C in series and X_2 is C.

10-6. Repeat Prob. 10-2 for a section in which X_1 is C and X_2 is L and C in parallel.

10-7. Repeat Prob. 10-2 for a section in which X_1 is C and X_2 is L and C in series.

10-8. Repeat Prob. 10-2 for a section in which X_1 is L and C in series and X_2 is also L and C in series but with a different resonant frequency.

10-9. Repeat Prob. 10-2 for a section in which X_1 is L and C in parallel and X_2 is L and C in series.

10-10. Repeat Prob. 10-2 for a section in which X_1 is L and C in series and X_2 is L.

10-11. Repeat Prob. 10-2 for a section in which X_1 is L and X_2 is L and C in parallel.

10-12. Repeat Prob. 10-2 for a section in which X_1 is L and C in parallel and X_2 is also L and C in parallel but with a higher resonant frequency.

10-13. Repeat Prob. 10-2 for a section in which X_1 is L and C in parallel and X_2 is also L and C in parallel but with a lower resonant frequency.

10-14. Repeat Prob. 10-2 for a section in which X_1 is L and C in series and X_2 consists of a parallel combination of L and C in series with a series combination of L and C. The frequency at which $X_1 = 0$ is the frequency at which $X_2 = \infty$. (This is the most general *m*-derived band-pass T section.)

10-15. Repeat Prob. 10-2 for a section in which X_1 consists of a parallel combination of L and C in series with a different parallel combination of L and C, and X_2 is a parallel combination of L and C. The frequency at which $X_1 = 0$ is the frequency at which $X_2 = \infty$. (This is the most general *m*-derived band-pass Π section).

10-16. Show that Z_0 of a Π section is imaginary in the attenuation band.

10-17. Compute numerical values to check the curve of Fig. 10-17 at the points $f/f_c = 0.5, 1.1, 1.2, 2.0$, and 4.0.

10-18. Compute and plot a curve of image impedance similar to Fig. 10-17 but for a low-pass *m*-derived filter with terminal half-Π sections for which $m = 0.6$.

10-19. Find L_k and C_k (similar to Eq. 10-47) for a high-pass Π section.

10-20. Compute and plot a curve of attenuation in a band-pass section as in Fig. 10-19, but giving a numerical scale for attenuation in decibels (see Fig. 10-8).

10-21. Design a low-pass filter with cutoff at 1,000 cycles, to match 700-ohm input and output circuits. Use Π-type intermediate sections. Above 1,100 cycles, attenuation is to be at least 20 db.

10-22. Design a high-pass filter with cutoff at 1,000 cycles, to match 700-ohm input and output circuits. Use T-type intermediate sections. Below 900 cycles, attenuation is to be at least 20 db.

10-23. Add to the filter of Prob. 10-21 a section to provide "infinite" attenuation at 1,050 cycles, and plot attenuation as a function of frequency for the improved filter.

10-24. Add to the filter of Prob. 10-22 a section to provide "infinite" attenuation at 950 cycles, and plot attenuation as a function of frequency for the improved filter.

10-25. Show that the input impedance to a lattice network (Fig. 10-24) is $\sqrt{Z_a Z_b}$ (the image impedance) provided an impedance equal to $\sqrt{Z_a Z_b}$ is connected at the output terminals.

10-26. Design a lattice network to be equivalent to the T section of Fig. 10-1.

10-27. Using the equation for lattice networks, find the image impedance of the network designed in Prob. 10-26, at frequencies of 1,200 and 2,400 cycles.

10-28. Using equations for lattice networks, find the attenuation in decibels produced at 700 cycles by the network designed in Prob. 10-26.

10-29. Design a "square" attenuating pad to produce 6.0 db attenuation at all frequencies and to match a 700-ohm line on each side (see Chap. 9).

10-30. Design an H attenuating pad to produce 4.5 db attenuation at all frequencies and to match a 650-ohm line on each side (see Chap. 9).

CHAPTER 11

APPLICATION TO POWER LINES

1. Characteristics of Power Transmission. Efficiency is one of the major concerns of the power engineer. He has a commodity to transport, and it is necessary that as much as possible of this commodity reach the receiving end of the transmission line. The commodity of the power engineer is electric energy whereas, in a sense, that of the communication engineer is information, and this distinction accounts for much of their difference in point of view while using the same transmission-line theory.

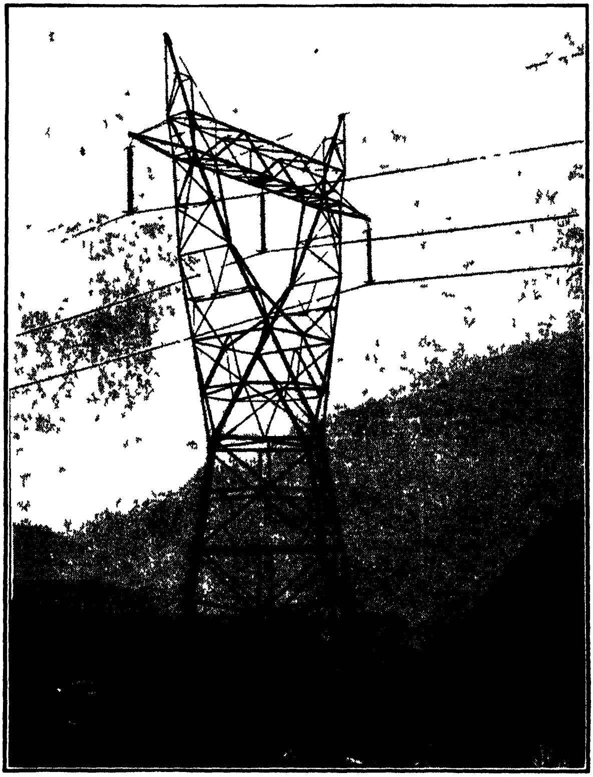
Several characteristics of power lines determine the mode of thought of the power engineer. All power lines are electrically short. A power line must have high efficiency. Voltage regulation is important for normal operation, and perhaps even more important for stability at time of fault. Terminal generators and transformers, and sometimes the line itself, have current-carrying capacities set by allowable heating which may not safely be exceeded. These factors, all peculiar to power transmission, have led to the development of special concepts and certain special tools. Among these are:

1. The concept of the flow of active power and reactive power.
2. Use of equivalent Π (or T) networks.
3. The "general circuit constants."
4. Circle diagrams and charts.

The power engineer usually handles a transmission-line problem in one of two ways. In his analysis of a power system, he may replace an actual transmission line by its equivalent Π or T section of lumped constants; he can then proceed with the simplification or transformation of his power network, or with its representation on a "network analyzer" without having to give further consideration to the complications introduced by distributed constants. Alternatively, he may use a graphic solution of the transmission-line equations, the *circle diagram* or *circle chart* of a power transmission line, and thus base the analysis of his network on the line equations or (what is really the same thing) on the *general circuit constants* of the line.

Both the computation of equivalent Π or T networks and the use of circle diagrams will be discussed in the following chapter.

2. Real and Reactive Power. Electrical behavior of a circuit may be expressed in terms of voltage, current, and power factor. The power engineer, however, more commonly works with voltage, active power, and reactive power. These quantities are indicated on switchboard instruments, and are convenient for thinking and speaking about power systems. Current, power factor, and other electrical quantities can be derived from voltage, active power, and reactive power if needed. The concept of reactive power is used very little by communication engineers, but it is a great convenience in the consideration of power transmission.



A tower of a 287,000-volt power line, Hoover Dam to Los Angeles, above the Cajon Pass. Three segmental conductors, 1 40-inch diameter, 512,000-circular-mil copper, are spaced 390 inches, see Fig 5-3. Two overhead ground wires protect from lightning. (*Courtesy of General Cable Corporation.*)

Reactive power is the product of voltage, current, and the sine of the phase angle, as active power is the product of voltage, current, and the cosine of the phase angle. Active power is produced by a generator, and is consumed by a resistive load, or by a motor, or by the resistance of a transmission line. Reactive power is produced by an overexcited synchronous condenser, or by an overexcited generator, and is consumed by an inductive reactive load, by an induction motor, or by the inductive reactance of a transmission line.[1]

Active power flows from generator to load, being wasted to some extent in resistance loss along the way. A similar stream of reactive power is visualized, flowing from overexcited generators to inductive loads and being consumed in the inductive reactance of transformers and transmission lines.

Conservation of energy requires that there be a balance of active power produced and active power consumed. In the same way, there is a balance of reactive power produced and reactive power consumed. The more usual reactive elements are:

Producing Reactive Power	*Consuming Reactive Power*
Overexcited synchronous machines	Underexcited synchronous machines
Capacitive loads, including static capacitors	Inductive reactive loads
Distributed capacitance of transmission lines	Induction motors
	Reactance of transformers
	Reactance of transmission lines

A synchronous machine may either produce or consume reactive power. If two machines on the same system are equally excited (so that both operate at unity power factor), neither produces or consumes reactive power. But if the excitation of one is higher than that of the other, reactive power flows from the overexcited to the underexcited machine. This is true regardless of whether one or the other of the machines is a generator, a motor, or a synchronous condenser.

As a very rough first approximation, the stream of active power through a transmission system and the stream of reactive power through the same system are more or less independent of each other. Thus, Fig. 11-1 shows active power flowing from a generator to a load, while reactive power flows each way from an overexcited synchronous condenser at an intermediate substation.

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[1] The algebraic sign of reactive power has been a subject of debate for the past twenty-five years. If an inductive load consumes reactive power, a capacitive load may be looked upon as a source of reactive power. The alternative proposal is to reverse the signs and assume that an inductive element such as an induction motor acts as a source of reactive power, and that reactive power is consumed in capacitance or in an overexcited generator. It appears that most power engineers prefer and actually use the former concept, and it will be adopted in the present discussion.

The opposite convention (assuming a capacitive load to consume reactive power) was adopted by the American Institute of Electrical Engineers and approved by the American Standards Association in 1941 (see Ref. 1). However, the subject was reopened and a reversal of sign was recommended by the American Institute of Electrical Engineers Standards Committee; the recommendation, with reasons and discussion, was published in *Electrical Engineering*, November, 1946, p. 512. It appears highly probable that the latter convention will continue to be commonly accepted by practicing engineers, and for that reason it is used in this book.

The flow of either active or reactive power through a transmission line results in consumption of real power in the resistance of the line, and the consumption of reactive power in the inductive reactance of the line. Thus the circulation of reactive power results in consumption of active power, and the circulation of active power results in consumption of reactive power. Obviously, active and reactive power are not entirely independent.

A transmission line produces reactive power in its shunt capacitance, and consumes reactive power in its series inductance. The amount produced is dependent on voltage, and the amount consumed is dependent on current.

There is a critical amount of load that makes the consumption of reactive power equal to production of reactive power in the line. It is not surprising to find that this balanced condition results if the impedance of the load is approximately the characteristic impedance of the line (exactly so if the line is a distortionless or lossless line). Also, the balance is approximated with any load if the line is a multiple of a half-wavelength long, but this is not a practical consideration on any existing power line.

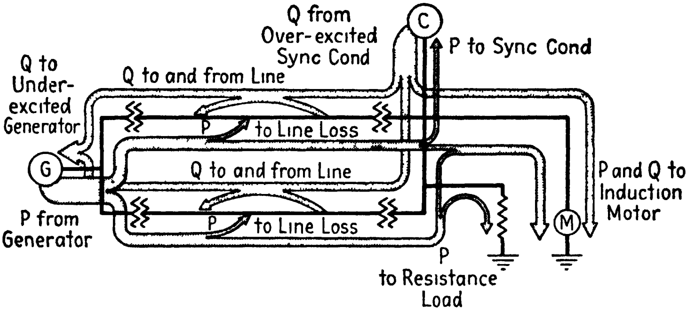

FIG. 11-1. Flow charts of active and reactive power in a simple transmission system.

In general, for good efficiency and good voltage regulation, reactive power should be produced as near as possible to the point at which it is consumed. This is an important reason for the use of synchronous or static condensers at the load end of a transmission line. Induction motors are usually the chief consumers of reactive power.

Reactive power is measured in *vars* (from the initials of *volt-amperes reactive*) as active power is measured in watts. In practical quantities, the units are more commonly kilovars or megavars, corresponding to kilowatts and megawatts. The algebraic symbol Q is used, as P is used for power.

From the definitions of P and Q as the product of voltage and current times the cosine and sine, respectively, of the phase angle, we may write

$$V\bar{I} = P + jQ \qquad (11\text{-}1)$$

where $\bar{I}$ is the conjugate of I, the complex quantity representing current.[2] This equation is the starting point of most studies of power.

3. Equivalent Π or T Network. A power transmission line is usually part of a fairly complex system, and the operation of the system must be studied as a whole. At the least, there are transformers at each end of a line, generators at one or both ends, and various kinds of loads. Often the most convenient way to include a transmission line in a general system problem is to substitute for the line an equivalent Π or T network, an artificial line (see Fig. 11-2). The elements of the artificial line can then be combined in series or parallel with impedances and admittances of transformers, generators, and other equipment constituting the power network.

To help understand the problem, let us consider the *network analyzer* or *network calculating board* by which power system problems are often studied. This is essentially a miniature network of adjustable impedances. Its impedances are set to be proportional to the impedances of transformers, generators, and other elements of the power network being studied. Voltage is applied at appropriate points to represent generator voltages. Resistances and reactances represent loads. The calculating-board voltages and currents are proportional to those actually existing in the power system, but reduced in scale by a factor that may be as much as 100,000 to 1. In physical appearance, the network analyzer resembles a rather extensive switchboard with panel after panel of adjustable knobs and dials, a few panels of jacks and plugs like a telephone exchange, and a central panel or bench equipped with various instruments for measuring voltage, current, and active and reactive power.

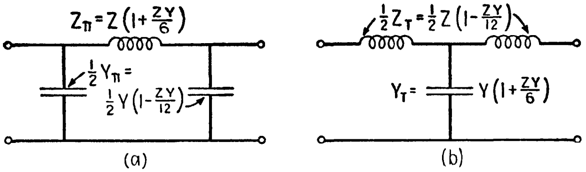
$Z_\pi = Z(1+\frac{ZY}{6})$
$\frac{1}{2}Y_\pi = \frac{1}{2}Y(1-\frac{ZY}{12})$
(a)
$\frac{1}{2}Z_T = \frac{1}{2}Z(1-\frac{ZY}{12})$
$Y_T = Y(1+\frac{ZY}{6})$
(b)

FIG. 11-2. Transmission-line equivalents. Π and T networks are shown for a line for which $zx = Z$ and $yx = Y$. Formulas given are obtained by series expansion, neglecting terms of second and higher degree. The general circuit constants of the line are, with the same approximation,

$$A = D = 1 + \tfrac{1}{2}ZY$$
$$B = Z(1 + \tfrac{1}{6}ZY)$$
$$C = Y(1 + \tfrac{1}{6}ZY)$$

See also Fig. 11-4.

A transmission line can be represented on a calculating board by the Π combination of impedances shown in Fig. 11-2*a*, or by the T combination of Fig.

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[2] As part of the disagreement previously mentioned, many authorities prefer $\bar{V}I = P + jQ$. This results in changing the sign of Q (see footnote 1). Common practice favors Eq. 11-1.

11-2*b*. Whether to use a T or Π is a matter of convenience; usually but not always, the Π section is more suitable for use on a calculating board. The Π or T section is, of course, an artificial line, as was discussed in Chap. 8.

Although the network analyzer or calculating board has been mentioned, it is not necessary to have such a device to solve network problems. With sufficient time and patience, a solution can always be obtained by computation; with simple networks, computation may be the preferred method. In either case, whether the desired values are to be measured in the miniature system of a network analyzer, or to be computed with pencil and paper, the transmission line may be visualized as a Π (or T) artificial-line section.

Our immediate interest is therefore in the representation of a transmission line by a Π or T section. The operation of a network analyzer, or the analysis of a power network by calculation, are subjects outside the scope of this book.

Artificial-line Formulas. Figure 11-2*a* shows a Π-section artificial line. For the Π section to be an exact representation of the real transmission line, the conditions of Chap. 8 must be satisfied. With a change of notation to fit customary usage, the following equations may be taken from Table 8-1:

$$Z_\pi = Z\frac{\sin Bx}{Bx} \tag{11-2}$$

$$\frac{1}{2}Y_\pi = \frac{1}{2}Y\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} \tag{11-3}$$

where Z_π is the series impedance in the equivalent Π network

$\frac{1}{2}Y_\pi$ is the admittance of each of the shunt branches of the Π

Z is the impedance per unit length of the actual line multiplied by the length of the line; that is, $Z = zx$

Y is the admittance per unit length of the actual line multiplied by the length of the line; that is, $Y = yx$

Bx is (as in previous chapters) equal to $-j\sqrt{zy}\,x$

Note that the fractions in Eqs. 11-2 and 11-3 are the *correction factors* of Chap. 8, and approach 1 for short lines.

For actual use, it is usually more convenient to put these correction factors in series form,[3] giving

$$\frac{\sin Bx}{Bx} = 1 + \frac{ZY}{6} + \frac{(ZY)^2}{120} + \frac{(ZY)^3}{5{,}040} + \cdots \tag{11-4}$$

$$\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} = 1 - \frac{ZY}{12} + \frac{(ZY)^2}{120} - \frac{17(ZY)^3}{20{,}160} + \cdots \tag{11-5}$$

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[3] Derivation of Eq. 11-4 follows:

$$\frac{\sin Bx}{Bx} = 1 - \frac{(Bx)^2}{3!} + \frac{(Bx)^4}{5!} - \frac{(Bx)^6}{7!} + \cdots$$

$$(Bx)^2 = -(zy)x^2 = -(zx)(yx) = -ZY$$

Elements of the equivalent Π *network* are then

$$Z_\pi = Z\left(1 + \frac{ZY}{6} + \frac{(ZY)^2}{120} + \frac{(ZY)^3}{5{,}040} + \cdots\right) \tag{11-6}$$

$$\frac{1}{2}Y_\pi = \frac{1}{2}Y\left(1 - \frac{ZY}{12} + \frac{(ZY)^2}{120} - \frac{17(ZY)^3}{20{,}160} + \cdots\right) \tag{11-7}$$

The advantage of the series form arises from the fact that all power lines are electrically short. It is practically never necessary to go beyond the first-order terms in the expansions of the correction factors to obtain sufficient accuracy. The expressions shown in Fig. 11-2*a* then result, and these are the *practical expressions to be used in almost all cases.*

For a power line of ordinary dimensions, 300 miles long, and operating at a 60-cycle frequency, the error in the simple expressions of Fig. 11-2 is between ½ and 1 per cent. Since there are usually other sources of error of greater magnitude, these simple expressions are commonly quite satisfactory. Moreover, the error is greatly less for shorter lines (error is approximately proportional to the fourth power of x).

Elements of the equivalent T *network* can also be taken from Table 8-1:

$$\frac{1}{2}Z_T = \frac{1}{2}Z\frac{\tan \frac{1}{2}Bx}{\frac{1}{2}Bx} \tag{11-8}$$

$$Y_T = Y\frac{\sin Bx}{Bx} \tag{11-9}$$

where Y_T is the shunt admittance in the equivalent T network
$\frac{1}{2}Z_T$ is the impedance of each of the series arms of the T
Z, Y, B, x are as tabulated under Eq. 11-3.

Using the series expansions of Eqs. 11-4 and 11-5, these are expressed in more easily usable form as

$$\frac{1}{2}Z_T = \frac{1}{2}Z\left(1 - \frac{ZY}{12} + \frac{(ZY)^2}{120} - \frac{17(ZY)^3}{20{,}160} + \cdots\right) \tag{11-10}$$

$$Y_T = Y\left(1 + \frac{ZY}{6} + \frac{(ZY)^2}{120} + \frac{(ZY)^3}{5{,}040} + \cdots\right) \tag{11-11}$$

As with the Π network, terms in the expansion of higher than first order can be neglected in most computations on power lines. Omitting higher order terms, we obtain the expressions of Fig. 11-2*b* which are *adequate for almost all practical computation.*

Substitution of the latter in the former gives Eq. 11-4. Equation 11-5 is derived in a similar manner from the formula

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots$$

Alternative Means of Computation. The correction factors are expanded in Eqs. 11-4 and 11-5 into infinite series for easy computation. This is not the only means of computation of the factors, however, and two alternatives should be mentioned.

1. The correction factors, each of which involves a trigonometric function of a complex quantity in the numerator, can be evaluated by expansion into expressions containing circular and hyperbolic functions of real arguments only, using Formulas 24 and 25*a*, respectively, of Table II. This is not usually the easiest method for power transmission problems.

2. The correction factors can be evaluated by the use of charts. Charts of the complex functions, such as Fig. 4-5 or Kennelly's charts (Ref. 12), can be used, but special charts have been prepared by Woodruff that are specifically for power lines and cover the usual useful region with a greatly enlarged scale and a convenient arrangement. Woodruff's charts[4] are for $\cosh \theta$, $(\sinh \theta)/\theta$, and $(\tanh \frac{1}{2}\theta)/\frac{1}{2}\theta$. Woodruff specifies that $\theta = \sqrt{ZY}$, Z and Y having the same meaning as in this chapter; so it will be recognized that the latter two charts give the correction factors of Eqs. 11-4 and 11-5. (This hyperbolic form of the correction factors is given explicitly in Table 8-1.)

Some power engineers prefer the use of Woodruff's charts; others prefer the expressions of Fig. 11-2, derived from series. For either method of computation, it is first necessary to calculate ZY. In most power-line computations, the conductance g that represents leakage and insulation loss can be neglected, in which case

$$ZY = (R + jX)(jB) = -BX + jBR \tag{11-14}$$

where B is susceptance[5] of the total line, found by multiplying the suscept-

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[4] Published in *Electrical Engineering* (*Transactions of the American Institute of Electrical Engineers*), May, 1935, and also in books by Woodruff and Kimbark (Refs. 38 and 14). It should be mentioned that the first two of Woodruff's charts can be used to evaluate the complex functions in the transmission-line equations by changing the form of Eqs. 4-32 and 4-34 to

$$V = V_r \cosh \theta + I_r Z \frac{\sinh \theta}{\theta} \tag{11-12}$$

$$I = I_r \cosh \theta + V_r Y \frac{\sinh \theta}{\theta} \tag{11-13}$$

recognizing that $Z_0 = Z/\theta = \theta/Y$, since by definition $\theta = \sqrt{ZY}$.

[5] It is regrettable that the symbol B is used with three different meanings in this chapter. Common usage is so well established that a certain amount of overlapping of meaning cannot readily be avoided. Here B is susceptance. In the next section, B is the second of the general circuit constants $ABCD$. In Eqs. 11-2 to 11-9, $B = -j\sqrt{zy}$, the generalized form of the phase constant β, as in the earlier chapters of this book. Fortunately, the meaning of B in any case can readily be determined from the context.

ance per mile, the reciprocal of $x_a' + x_d'$ from the tables of Appendix 3, by the length of the line (capacitive susceptance is positive)

X is reactance of the total line, found by multiplying the reactance per mile, from the tables of Appendix 3, by the length of the line

R is resistance of the total line, found by multiplying the resistance of one conductor per mile, from the tables of Appendix 3, by the length of the line

This expression gives ZY in rectangular complex form, which makes its use in the formulas of Fig. 11-2 particularly simple.

The behavior of a power system can be fully worked out by representing each transmission line by an equivalent Π or T network. An equivalent Π or T network does not, of course, give any information about conditions along the line itself, but its effect on all the rest of the power system is identical with that of the actual line. The equivalence is exact at one frequency only, but a power line is operated at only one frequency.

After equivalent Π or T sections are substituted for transmission lines in a power network, the solution for power-system behavior is carried forward either by computation or by use of a network calculating board.

4. General Circuit Constants. Another useful method of analysis of power-line operation is provided by the *general circuit constants* usually written *ABCD*. These constants are defined by their appearance in the following equations.

For a network (passive, linear, and bilateral) having two pairs of terminals, such as the input and output terminals of a transmission line, the input voltage and current can be written in terms of the output voltage and current and four constants:

$$V_s = AV_r + BI_r \qquad (11\text{-}15)$$

$$I_s = CV_r + DI_r \qquad (11\text{-}16)$$

The constants *ABCD* are determined by the network; indeed, they completely define the terminal behavior of the network, and two networks having identical constants *ABCD* are equivalent as far as their terminal current and voltage relations are concerned.[6]

Examples. A transmission line is an example of a network for which the constants *ABCD* are readily determined. Let us write Eq. 4-9 as follows, letting x be the total length of the line:

$$V_s = V_r \cos(\beta - j\alpha)x + jI_rZ_0 \sin(\beta - j\alpha)x \qquad (11\text{-}17)$$

---

[6] The matrix $\begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$ can be used to characterize the network. This leads to interesting mathematical methods for those familiar with matrices (see footnotes 8 and 9).

Then, by comparison with Eq. 11-15, it is evident that for a *transmission line*:

$$A = \cos(\beta - j\alpha)x \tag{11-18}$$

$$B = jZ_0 \sin(\beta - j\alpha)x \tag{11-19}$$

Equation 4-13 gives

$$I_s = j\frac{V_r}{Z_0}\sin(\beta - j\alpha)x + I_r\cos(\beta - j\alpha)x \tag{11-20}$$

and comparison with Eq. 11-16 shows that for a *transmission line*:

$$C = \frac{j}{Z_0}\sin(\beta - j\alpha)x \tag{11-21}$$

$$D = \cos(\beta - j\alpha)x \tag{11-22}$$

In the above equations, the constants $ABCD$ of a transmission line are expressed in circular trigonometric functions. Hyperbolic functions can be used instead, if preferred, and from Eqs. 4-32 and 4-34 the following constants are obtained:

$$A = \cosh(\alpha + j\beta)x \tag{11-23}$$

$$B = Z_0 \sinh(\alpha + j\beta)x \tag{11-24}$$

$$C = \frac{1}{Z_0}\sinh(\alpha + j\beta)x \tag{11-25}$$

$$D = \cosh(\alpha + j\beta)x \tag{11-26}$$

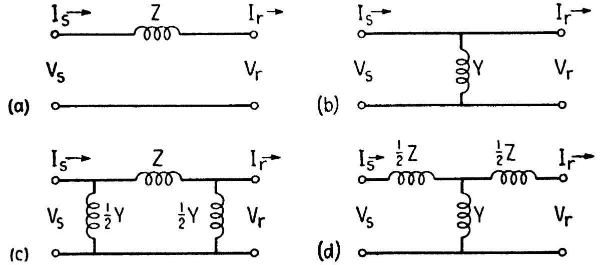

FIG. 11-3. General circuit constants of various network elements. See Eqs. 11-29, 11-32, 11-36, and 11-39.

The general circuit constants of various simple network elements are easily obtained by similar comparison. For instance, for the series impedance of Fig. 11-3*a*, we may write

$$V_s = V_r + ZI_r \tag{11-27}$$

$$I_s = I_r \tag{11-28}$$

and comparison with the general Eqs. 11-15 and 11-16 gives for the *series impedance*:

$$A = 1 \qquad B = Z$$

$$C = 0 \qquad D = 1 \tag{11-29}$$

From Fig. 11-3*b*,

$$V_s = V_r \tag{11-30}$$

$$I_s = I_r + YV_r \tag{11-31}$$

and these indicate that for the simple *shunt admittance*

$$A = 1 \qquad B = 0$$

$$C = Y \qquad D = 1 \tag{11-32}$$

For the symmetrical Π of Fig. 11-3*c*:

$$\begin{aligned} V_s &= V_r + Z(I_r + \tfrac{1}{2}YV_r) \\ &= V_r(1 + \tfrac{1}{2}ZY) + I_rZ \end{aligned} \tag{11-33}$$

$$I_s = I_r + \tfrac{1}{2}YV_r + \tfrac{1}{2}YV_s \tag{11-34}$$

Substituting Eq. 11-33 for V_s and rearranging terms gives

$$I_s = V_r(Y + \tfrac{1}{4}ZY^2) + I_r(1 + \tfrac{1}{2}ZY) \tag{11-35}$$

Hence the general circuit constants of the *symmetrical* Π *network* are

$$A = 1 + \tfrac{1}{2}ZY \qquad B = Z$$

$$C = Y + \tfrac{1}{4}ZY^2 \qquad D = 1 + \tfrac{1}{2}ZY \tag{11-36}$$

For the T network of Fig. 11-3*d*:

$$V_s = V_r + \tfrac{1}{2}ZI_r + \tfrac{1}{2}ZI_s \tag{11-37}$$

$$I_s = I_r + Y(V_r + \tfrac{1}{2}ZI_r) \tag{11-38}$$

The latter of these must be substituted into the former to eliminate I_s, and when terms are collected we see that the general circuit constants of the *symmetrical* T *network* are

$$A = 1 + \tfrac{1}{2}ZY \qquad B = Z + \tfrac{1}{4}Z^2Y$$

$$C = Y \qquad D = 1 + \tfrac{1}{2}ZY \tag{11-39}$$

It is sometimes useful to have, in addition to the above, the constants of unsymmetrical Π and T networks. These are given in Fig. 11-4. Such examples could be continued indefinitely. More complicated networks can be considered to be combinations of those already treated, and the constants *ABCD* for a complex network can be computed from the constants that apply to its component parts individually, using some of the following general relations.

General Relations. 1. For any network,[7] the following relation between the constants is always true:

$$AD - BC = 1 \tag{11-40}$$

Illustrations of this will be seen in the general circuit constants of the five examples given above. To see that it is true in general, it may be shown to be true for an unsymmetrical Π or T; then since any network of the kind we are considering can be simplified to an equivalent Π or T, Eq. 11-40 must hold for all such networks. (A more formal proof is obtained by expressing the constants *ABCD* in terms of driving point and transfer impedances of a general network.)

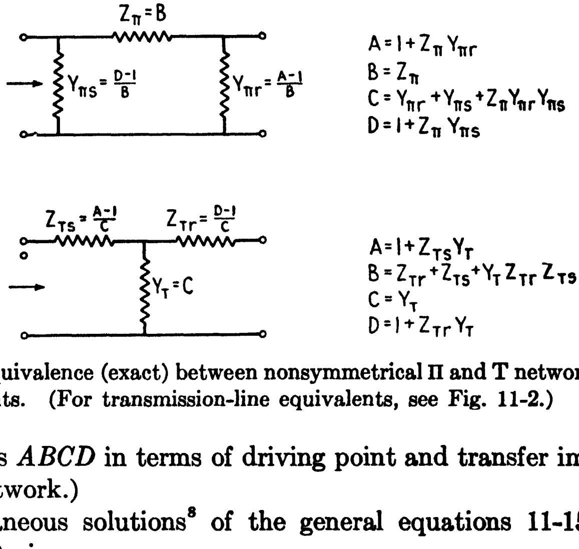
$Z_\pi = B$
$Y_{\pi s} = \frac{D-1}{B}$
$Y_{\pi r} = \frac{A-1}{B}$
$A = 1 + Z_\pi Y_{\pi r}$
$B = Z_\pi$
$C = Y_{\pi r} + Y_{\pi s} + Z_\pi Y_{\pi r} Y_{\pi s}$
$D = 1 + Z_\pi Y_{\pi s}$
$Z_{Ts} = \frac{A-1}{C}$
$Z_{Tr} = \frac{D-1}{C}$
$Y_T = C$
$A = 1 + Z_{Ts} Y_T$
$B = Z_{Tr} + Z_{Ts} + Y_T Z_{Tr} Z_{Ts}$
$C = Y_T$
$D = 1 + Z_{Tr} Y_T$

FIG. 11-4. Equivalence (exact) between nonsymmetrical Π and T networks and general circuit constants. (For transmission-line equivalents, see Fig. 11-2.)

2. Simultaneous solutions[8] of the general equations 11-15 and 11-16 (using 11-40) gives

$$V_r = DV_s - BI_s \tag{11-41}$$

$$I_r = -CV_s + AI_s \tag{11-42}$$

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[7] Passive, linear, bilateral, and with two pairs of terminals, as already stipulated.

[8] Using determinants, Eq. 11-40 is

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1 \tag{11-43}$$

This, with Eqs. 11-15 and 11-16, gives

$$V_r = \begin{vmatrix} V_s & B \\ I_s & D \end{vmatrix} \qquad I_r = \begin{vmatrix} A & V_s \\ C & I_s \end{vmatrix} \tag{11-44}$$

which are Eqs. 11-41 and 11-42.

3. A very useful relation is that $A = D$ for all *symmetrical* networks. A symmetrical network is one that looks the same from either end; that is, the sending and receiving terminals may be interchanged without changing the behavior of the network. This equality of A and D may be deduced from a comparison of Eqs. 11-41 and 11-42 with 11-15 and 11-16, remembering that the assumed positive direction of current flow is reversed when sending and receiving terminals are interchanged.

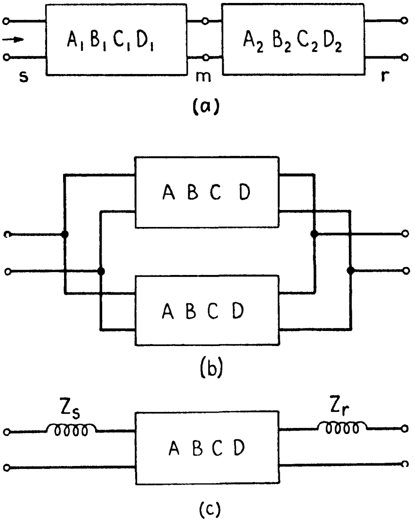

FIG. 11-5. General circuit constants of composite systems. See Eqs. 11-49, 11-50, and 11-51.

Networks in five of the above examples are symmetrical (the transmission line, simple impedance, simple admittance, symmetrical Π, and symmetrical T). We found that $A = D$ for all of these, but not for the unsymmetrical Π and T of Fig. 11-4.

4. If a network with constants $A_1B_1C_1D_1$ is connected in cascade with a network with constants $A_2B_2C_2D_2$, the constants $A_0B_0C_0D_0$ that apply to the entire composite network can be determined. Voltage and current between the two networks, at point m of Fig. 11-5*a*, can be written

$$V_m = A_2V_r + B_2I_r \tag{11-45}$$

$$I_m = C_2V_r + D_2I_r \tag{11-46}$$

Then sending voltage and current are

$$V_s = A_1V_m + B_1I_m = (A_1A_2 + B_1C_2)V_r + (A_1B_2 + B_1D_2)I_r \tag{11-47}$$

$$I_s = C_1V_m + D_1I_m = (C_1A_2 + D_1C_2)V_r + (C_1B_2 + D_1D_2)I_r \tag{11-48}$$

from which the constants of *two networks in cascade* are[9]

$$\begin{aligned} A_0 &= A_1A_2 + B_1C_2 & B_0 &= A_1B_2 + B_1D_2 \\ C_0 &= C_1A_2 + D_1C_2 & D_0 &= C_1B_2 + D_1D_2 \end{aligned} \tag{11-49}$$

This set of constants for two networks in cascade is useful when, for example, it is desired to find $ABCD$ for a transmission line supplied through

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[9] Using matrices, the combination in cascade is written

$$\begin{Vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{Vmatrix} = \begin{Vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{Vmatrix} \cdot \begin{Vmatrix} A_2 & B_2 \\ C_2 & D_2 \end{Vmatrix}$$

a transformer, the constants of the transformer and of the line being known independently. By repeated application, constants can be found for a composite system of any number of networks in cascade. It is possible to derive explicit expressions for *ABCD* of three general networks in series, or four, or any number, but the complication of such expressions increases rapidly.

5. The constants *ABCD* for any two networks in parallel can be found, but as the most interesting case is that in which the two parallel networks are identical, as in Fig. 11-5*b*, the expressions will be given for that case only. For *two identical networks in parallel*, each having constants *ABCD*,

$$A_0 = A \qquad B_0 = \tfrac{1}{2}B$$
$$C_0 = 2C \qquad D_0 = D \tag{11-50}$$

Sets of constants can be thus derived for more or less special cases. (Many additional examples will be found in Refs. 5, 16, and 18. The Westinghouse "Reference Book," Ref. 18, may be specially mentioned.) Only one more example will be given here.

6. Most power transmission links consist of a transmission line with transformers at each end, and the composite system of Fig. 11-5*c* is frequently an adequate representation. Z_s and Z_r are here intended to represent the series impedances of the terminal transformers, and *ABCD* may be the constants of the intervening transmission line. The constants of a *network with a series impedance at each end* are

$$A_0 = A + CZ_s \qquad B_0 = B + AZ_r + DZ_s + CZ_sZ_r$$
$$C_0 = C \qquad D_0 = D + CZ_r \tag{11-51}$$

5. General Circuit Constants of Transmission Lines. The constants *ABCD* for a transmission line are complex quantities. They can be computed from Eqs. 11-18, 11-19, 11-21, and 11-22, or from Eqs. 11-23 to 11-26, by expressing these functions of complex variables as combinations of functions of real variables, or by use of charts.[10] However, the constants are readily expanded into infinite series, and good accuracy can easily be obtained for lines as long as 300 miles, if the frequency and general characteristics of the lines are in the range that is usual for power engineering, without going beyond the first-order terms of the series. This provides a convenient and practical means of computation.

As in Eq. 11-23,

$$A = \cosh \sqrt{zy}\,x = \cosh \sqrt{ZY} \tag{11-52}$$

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[10] The section headed Alternative Means of Computation, p. 273, applies to computation of the constants *ABCD* as well as to computation of equivalent Π and T networks.

where z is series impedance per unit length of line
y is shunt admittance per unit length of line
x is length of line
$Z = zx$ and $Y = yx$

Using Formula 5 of Table II, the hyperbolic cosine is expanded:

$$A = 1 + \frac{ZY}{2} + \frac{(ZY)^2}{24} + \frac{(ZY)^3}{720} + \cdots \tag{11-53}$$

Similar expansion of Eqs. 11-24 to 11-26, noting that $Z_0 = \sqrt{Z/Y}$, gives

$$B = Z\left(1 + \frac{ZY}{6} + \frac{(ZY)^2}{120} + \frac{(ZY)^3}{5{,}040} + \cdots\right) \tag{11-54}$$

$$C = Y\left(1 + \frac{ZY}{6} + \frac{(ZY)^2}{120} + \frac{(ZY)^3}{5{,}040} + \cdots\right) \tag{11-55}$$

$$D = A \tag{11-56}$$

As mentioned, for lines of usual dimensions, higher order terms may be neglected without reducing the accuracy that can be obtained from ordinary engineering data, and the general circuit constants of a *power transmission line are approximately*

$$A = 1 + \frac{ZY}{2} \qquad B = Z\left(1 + \frac{ZY}{6}\right)$$

$$C = Y\left(1 + \frac{ZY}{6}\right) \qquad D = A \tag{11-57}$$

For the value of ZY, see Eq. 11-14. These equivalents are given with Fig. 11-2.

It is sometimes helpful to change an actual transmission line to an equivalent Π or T, and then (perhaps after combining with other network impedances) to find the network constants $ABCD$ of the Π or T. For a symmetrical network, Eqs. 11-36 or 11-39 can be used; for a nonsymmetrical network, the equations are given with Fig. 11-4.

The inverse transformation should also be available: if the constants $ABCD$ of a transmission line (or other network) are known, what is the Π or T equivalent? For a *symmetrical network the elements of the equivalent* Π are

$$Z_\pi = B \qquad \frac{1}{2}Y_\pi = \frac{A - 1}{B} \tag{11-58}$$

These are obtained by solving Eqs. 11-36 for Z and $\frac{1}{2}Y$. Similarly, for a *symmetrical network, the elements of the equivalent* T are

$$Y_T = C \qquad \frac{1}{2}Z_T = \frac{A - 1}{C} \tag{11-59}$$

These are derived from Eqs. 11-39. The derivations are similar for unsymmetrical networks, and results are given in Fig. 11-4.

The general circuit constants of transmission systems are sometimes convenient for computation, and they are used in preparing circle diagrams of transmission lines.

6. The Transmission-line Vector Diagram. It is evident that a vector diagram can be drawn for a transmission line, as in Fig. 11-6*a*, using the general circuit constants of Eq. 11-15,

$$V_s = AV_r + BI_r \tag{11-60}$$

It will be convenient to write the general circuit constants in both polar and rectangular form, using the following notation:

$$A = |A|\,\underline{/\alpha} = a_1 + ja_2$$

$$B = |B|\,\underline{/\beta} = b_1 + jb_2$$

$$C = |C|\,\underline{/\gamma} = c_1 + jc_2$$

$$D = |D|\,\underline{/\delta} = d_1 + jd_2 \qquad (11\text{-}61)$$

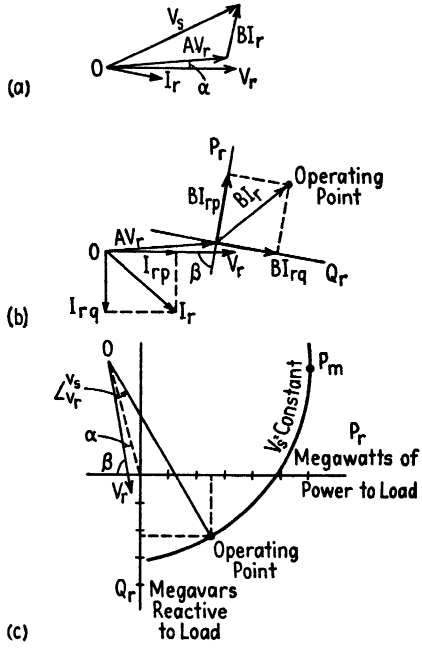

FIG. 11-6. Evolution of the receiving-end circle diagram.

For an ordinary power transmission line, A (which is cosh $\sqrt{zy}\,x$) is a complex quantity with a magnitude slightly less than 1 and a small positive angle. The angle, α, is shown in Fig. 11-6*a*. B for the ordinary line has an angle β that is nearly 90 degrees. If receiving-end current is, as shown in the figure, substantially in phase with receiving-end voltage, BI_r is at a large leading angle with respect to AV_r. The sum of AV_r and BI_r is V_s. It may be noted that if there is no load on the line, $V_s = AV_r$.

To develop a circle diagram, consider that V_r is held constant, and that V_s is to be found as the load varies. Changing the load current I_r changes the vector BI_r. The vector V_s changes correspondingly.

It is convenient to consider the in-phase and quadrature components of current separately. In Fig. 11-6*b*, I_{rp} is the component of I_r that is in phase with the receiving-end voltage, and I_{rq} is the component that lags the voltage by 90 degrees. Any receiving-end current may thus be analyzed into components, and each component is then multiplied by B. BI_{rp} is at an angle β with V_r, and BI_{rq} is normal to BI_{rp}.

Since I_{rp} is the in-phase component of receiving-end current, and receiving-end voltage is constant, BI_{rp} is proportional to received power. Similarly, BI_{rq} is proportional to received reactive.[11] This proportionality makes it possible to draw a pair of axes, one coinciding with the line BI_{rp} and the other with BI_{rq}, and to calibrate them in megawatts and megavars, respectively. On axes so calibrated, values of received active and reactive power can be read if the *operating point*, the tip of the vector BI_r, is known. Conversely, if P_r and Q_r are known, the operating point on the diagram can be found. This is a convenience, for P_r and Q_r are more commonly known than are the components of the load current.

7. The Receiving-end Circle Diagram. By tipping the vector diagram of Fig. 11-6*b* to make the P_r axis horizontal and the Q_r axis vertical, Fig. 11-6*c* is obtained. This is a simple form of the transmission-line circle diagram. Active and reactive power supplied by the line to a load are read on its axes. An operating point is thereby located on the diagram, and the distance from the operating point to O is a measure of the sending-end voltage that is necessary. The angle at O is the angle between the receiving-end voltage and the sending-end voltage.

Since the distance from O to the operating point is the magnitude of V_s, all operating points that lie on a circle with center at O have the same sending-end voltage. Such a circle is drawn in the diagram, and other circles can be drawn for other sending-end voltages.

Figure 11-6*c* is called a receiving-end diagram because its axes give receiving-end values of active and reactive power. This simple form of the diagram is useful in problems such as the following: if the receiving-end voltage of a transmission line is that for which the diagram was drawn, what is the sending-end voltage (magnitude and angle) when the load on the line has a specified value? The diagram will give the sending-end voltage for any value of load. Note that if the load consumes reactive power, as an inductive load will do, the operating point lies below the horizontal axis, and if the load produces reactive power, as does a capacitor or an overexcited synchronous machine, the operating point is above the axis.[12] Similarly, of course, if the load were to produce active power instead of consuming it, the operating point would lie to the left of the vertical axis. On some lines, the direction of flow of active power may reverse from time to time, and for such a line the operating point may fall on either side of the vertical axis.

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[11] The word *reactive* is commonly used as a noun by power engineers, meaning *reactive power*. This usage, although regrettable, will probably be accepted for lack of any other single word to convey the same meaning.

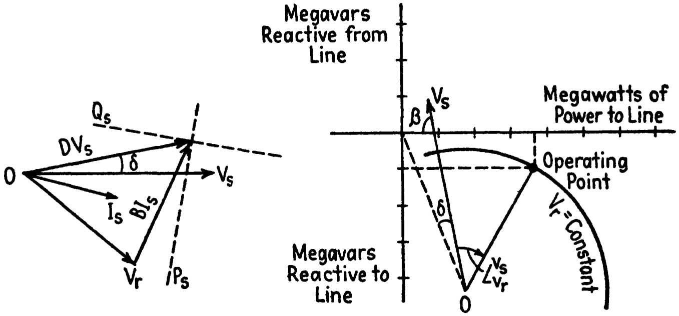
[12] Whether to use the circle diagram in this position, or to invert it so that the operating points of inductive loads lie above the power axis, has long been debated. The question is part of the controversy regarding the correct sign of reactive power. The orientation of Fig. 11-6 shows positive reactive power (according to the convention used in this book; that is, $V\bar{I} = P + jQ$) measured downward. Those who prefer to draw it upward may do so, thereby inverting the whole diagram. But

8. The Sending-end Circle Diagram. The circle diagram of Fig. 11-6 was developed from Eq. 11-60 for V_s. There is a similar equation 11-41 for V_r:

$$V_r = DV_s - BI_s \tag{11-62}$$

Figure 11-7*a* shows the relation of these quantities.

Consider the sending-end voltage to be constant. The vector DV_s is therefore fixed. If sending-end power changes, BI_s will change, and the tip of the vector V_r will move correspondingly. The two dash lines marked P_s and Q_s can be drawn as axes and, if they are properly calibrated in sending-end megawatts and sending-end megavars, respectively, it is possible to locate the lower end of the BI_s vector by reading on these axes the values of sending-end active and reactive power. The tip of the V_r vector is thereby determined, and from this the magnitude and angle of the receiving-end voltage are known.



(a) (b)

FIG. 11-7. The sending-end circle diagram.

This diagram may be turned as in Fig. 11-7*b*: it is called a sending-end diagram because the axes are calibrated in sending-end power and reactive power. A circle about *O*, such as the one shown, is a locus of operating points for which receiving-end voltage is constant. The angle indicated at

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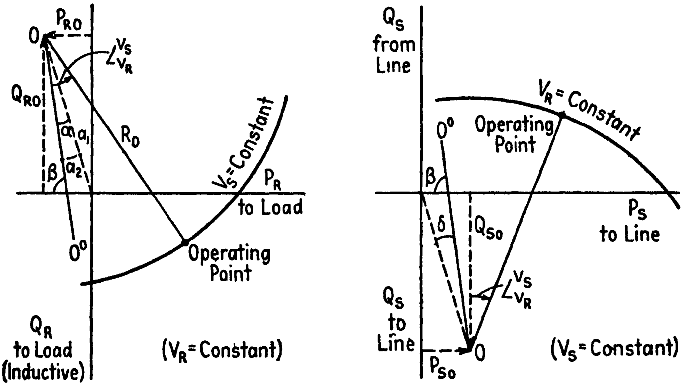
Fig. 11-6*c* follows so naturally from the vector diagram of Fig. 11-6*a* that it seems rather pedantic to insist on turning it upside down. The simple fact is that each engineer will draw his diagrams the way he likes—some right side up, some upside down, some sideways—and this is no doubt as it should be.

O is the angle between V_s and V_r; the reference vector V_s is at an angle β with the horizontal axis, and the angle δ is shown.

9. Construction of the Circle Diagrams. Circle diagrams can be drawn as suggested in Figs. 11-6 and 11-7. The necessary magnitudes and angles can be measured, and the scales of active and reactive power computed. However, work is saved, and chance of mistake is avoided, by systematizing the construction of the diagrams. For this purpose, certain dimensions of circle diagrams will now be derived and these dimensions may conveniently be used whenever a circle diagram is to be drawn.

Scales of active power P and reactive power Q are first laid out on a sheet of cross-section paper (refer to Fig. 11-8). These scales must be numerically equal in megawatts and megavars per inch on the paper, and should include the range of operating values of P and Q to be expected. The following information is then needed:



(a) Receiving End (b) Sending End

FIG. 11-8. Dimensions used in construction of circle diagrams.

1. The coordinates of point O on the cross-section paper.
2. The radii of constant-voltage circles.
3. The position of the reference line for zero angle between V_s and V_r.

To obtain this information for the *receiving-end diagram,* Eq. 11-60 is modified to the form of the equation of a circle. In Eq. 11-60, let V_r be a purely real reference vector. Then

$$\begin{aligned} V_s &= AV_r + BI_r \\ &= (a_1 + ja_2)V_r + (b_1 + jb_2)(I_{rp} + jI_{rq}) \qquad (11\text{-}63) \end{aligned}$$

$|V_s|^2$ is next set equal to the sum of the squares of the magnitudes of the real and imaginary components of V_s; terms are then rearranged, and by completing the squares of the quadratic expressions containing the components of current:

$$\left(I_{rp} + \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} V_r\right)^2 + \left(I_{rq} + \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2} V_r\right)^2 = \frac{|V_s|^2}{b_1^2 + b_2^2} \qquad (11\text{-}64)$$

Substituting into this equation the following:[13]

$$I_{rp} = \frac{P_r}{V_r} \qquad \text{and} \qquad I_{rq} = -\frac{Q_r}{V_r} \qquad (11\text{-}65)$$

and introducing the symbols P_R and Q_R (with capital subscripts) to represent total megawatts and megavars in the three phases of a balanced three-phase system (these being more convenient than watts and vars per phase), and letting V_S and V_R be the magnitudes of line-to-line voltage in kilovolts (rather than the line-to-neutral voltage in volts), Eq. 11-64 becomes

$$\left(P_R + \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} V_R^2\right)^2 + \left(Q_R - \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2} V_R^2\right)^2 = \frac{V_S^2 V_R^2}{b_1^2 + b_2^2} \qquad (11\text{-}66)$$

If V_S and V_R are constant, this equation has the familiar form of the equation of a circle, P_R and Q_R being the variables. In other words, Eq. 11-66 shows that the locus of operating points, plotted with P_R and Q_R as the axes, is a circle, provided sending-end voltage and receiving-end voltage are both held constant. This fact, of course, was already known, and such circles are shown in Figs. 11-6*c* and 11-8*a*. Now, however, we are prepared to continue with the construction of a circle diagram. If the coordinates of the center of a circle are called P_{RO} and Q_{RO}, and its radius R_O, as in Fig. 11-8*a*, Eq. 11-66 shows that

$$P_{RO} = -\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} V_R^2 \qquad (11\text{-}67)$$

$$Q_{RO} = +\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2} V_R^2 \qquad (11\text{-}68)$$

$$R_O = \frac{V_S V_R}{\sqrt{b_1^2 + b_2^2}} \qquad (11\text{-}69)$$

where P_{RO} and Q_{RO} are the coordinates of the center of the circle (measured in megawatts and megavars)

R_O is the radius of the circle (this value is measured on the megawatt scale of the P axis)

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[13] This derivation follows that of O. G. C. Dahl (Ref. 5). Equation 11-65 changes Dahl's choice of sign for reactive power. To be consistent with $V\bar{I} = P + jQ$, it is necessary to write $I_{rq} = -Q_r/V_r$ (see also footnote 14).

V_S and V_R are magnitudes of sending and receiving voltages, respectively, both being line-to-line values measured in kilovolts

P_R and Q_R are megawatts of total three-phase power, active and reactive, respectively, delivered by the line to the load

a_1, a_2, b_1, and b_2 are the real and imaginary components of the general circuit constants A and B, as in Eqs. 11-61, and may be computed from Eqs. 11-57.

If A and B include the impedance of terminal transformers and other apparatus, as do A_O and B_O of Eqs. 11-51, the circle diagram shows the over-all performance of the system consisting of the line and the transformers or other apparatus in series. Even synchronous-machine impedance is sometimes included in the determination of A and B, and hence in the diagram, but this does not usually give satisfactory results as the effect of machine saturation and the difference between direct and quadrature axis reactances are not readily taken into account.

With a power line of normal length and frequency, the components of A and B are all positive; hence P_{RO} is negative, as shown in Fig. 11-8*a*. Ordinarily, a_1 is much larger than a_2, and b_2 is much larger than b_1; Q_{RO} is therefore capacitive.[14]

To complete the receiving-end diagram, so that angles may be read from it as well as magnitudes, the reference line marked 0° in Fig. 11-8 must be located. As indicated in the figure, the angle of either of the general circuit constants, either α or β, will locate this reference line from which angles are to be measured. Good accuracy is more easily obtained by using α, and the simplest method is as follows. Draw the line from point O to the origin of coordinates. Measure from O, along this line, a distance proportional to a_1 (the real component of A), and thence a perpendicular distance proportional to a_2; the zero-degree line passes through the point thus determined.

To construct the *sending-end diagram,* Eq. 11-62 is modified and the following equation, similar to 11-66, is obtained:

$$\left(P_S - \frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2} V_S^2\right)^2 + \left(Q_S + \frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2} V_S^2\right)^2 = \frac{V_S^2 V_R^2}{b_1^2 + b_2^2} \quad (11\text{-}70)$$

From this equation, a circle of constant voltage is located, as shown in Fig. 11-8*b*, by

$$P_{SO} = \frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2} V_S^2 \quad (11\text{-}71)$$

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[14] With our choice of sign of reactive power, this is positive (see footnote 13). Whatever convention is used, if Eq. 11-68 gives a negative value, as it will for a normal power line, Q_{RO} is measured in the direction of reactive power delivered to a *capacitive* load.

$$Q_{SO} = -\frac{d_2 b_1 - d_1 b_2}{b_1^2 + b_2^2} V_S^2 \tag{11-72}$$

$$R_O = \frac{V_S V_R}{\sqrt{b_1^2 + b_2^2}} \tag{11-73}$$

Symbols have the meanings given below Eq. 11-69, and

P_{SO} and Q_{SO} are the coordinates of the center of the circle (measured in megawatts and megavars)

P_S and Q_S are megawatts of total three-phase power, active and reactive, respectively, delivered to the line

d_1 and d_2 are the real and imaginary components of the general circuit constant D. The angle of D is δ, and δ can be used as in Fig. 11-8*b* to locate the zero-degree reference line for measurement of angle

Note that the radius of a circle is the same in the sending-end and receiving-end diagrams. There is symmetry of location of the centers, also, if the line itself is symmetrical; that is, if $A = D$, making $a_1 = d_1$ and $a_2 = d_2$. (Indeed, with a symmetrical line, the sending-end diagram is identical with the left-hand half of the receiving-end diagram if proper attention is given to the subscripts S and R and the signs of P and Q.)

10. Examples. We may now outline the solution of a transmission-line problem. A receiving-end diagram and a sending-end diagram are used as a pair to give full information. Line constants A, B, and D are known. We also know the active and reactive power to the load, and the receiving-end voltage. Active and reactive power and voltage at the sending end, and line loss, are to be found.

P_R and Q_R scales are laid off on a sheet of cross-section paper. The center for constant-voltage circles is found, as in Fig. 11-8*a*, using Eqs. 11-67 and 11-68. The 0° line is drawn, a_1 and a_2 being known. The operating point is located from given values of active and reactive power.

The magnitude of V_S is first found by measuring the radius of a circle about O that passes through the operating point; Eq. 11-69 is used to compute the value of V_S. The angle between V_S and V_R is then found by measuring the angle from the zero-degree line to the operating point.

Next, the sending-end diagram is needed (see Fig. 11-8*b*). A separate piece of cross-section paper may be used. Since V_S is now known, from the receiving-end diagram, the center of the required circle on the sending-end diagram can be found (Eqs. 11-71 and 11-72). The radius of the required circle on the sending-end diagram is known; it is the same as the radius of the circle on the receiving-end diagram. The angle between V_S and V_R is known from the receiving-end diagram. A radial line is therefore drawn at the correct angle from the 0° line; a circular arc of proper radius is struck, and the point of intersection gives the operating point on the sending-end diagram.

Finally, sending-end active and reactive power are read from the chart as the coordinates of the operating point thus located. Power loss in the line was also to be found; it is the difference between P_S and P_R. Efficiency of transmission is P_R divided by P_S. Sending-end power factor is the cosine of the angle between the power axis and a line from the origin of coordinates to the operating point.

Many modifications are possible to permit small economies in the construction or use of circle diagrams. One of the most common is to superimpose the sending-end diagram on the receiving-end diagram, using the same axes for both. Thus, Fig. 11-8*b* can be superimposed on Fig. 11-8*a*, with the axes coinciding. This provides a direct comparison, on the same sheet of paper, of receiving-end P and Q with sending-end P and Q. Active power consumed in the line is given by the horizontal distance between the sending-end operating point and the receiving-end operating point, and reactive power consumed in the line is given by the vertical distance between the points. The disadvantage of such superposition of the diagrams is that the axes cannot readily be marked unambiguously; for instance, P_S and P_R are read on the same scale, and the beginner may be at a loss to interpret his results. After a little experience, however, this difficulty vanishes, and the advantages of the combined diagram usually justify its use.

A numerical example will illustrate certain points that have not previously appeared. Consider a transmission line (Oroville to Bellota) 111.5 miles long. The conductors, each 795,000-circular-mil ACSR cable with 54 aluminum and 7 steel strands, are spaced 15.88 feet apart in a vertical plane; the temperature of the conductors is taken to be 50°C, corresponding to moderately heavy load. The nominal voltage is 230 kilovolts at 60 cycles.

Computation gives $a_1 = 0.974$, $a_2 = 0.00475$, $b_1 = 15.15$, $b_2 = 84.5$.

Consider that the load on the line is 120 megawatts power and 40 megavars reactive (inductive) load. Receiving voltage is 230 kilovolts. Find sending-end voltage, and sending-end active and reactive power, using a circle diagram.

The first step is to compute the coordinates of the center of the circle on the receiving-end chart, using Eqs. 11-67 and 11-68. These are $P_{RO} = -109.1$ and $Q_{RO} = -590$.

Figure 11-9 is to be used for both receiving-end and sending-end diagrams. The center of the receiving-end circles is located in the upper left-hand corner using the computed values of P_{RO} and Q_{RO}. The zero-degree line, starting from point O, is drawn at an angle α to the dash line from O to the origin of coordinates; this is done by measuring 9.74 inches along the dash

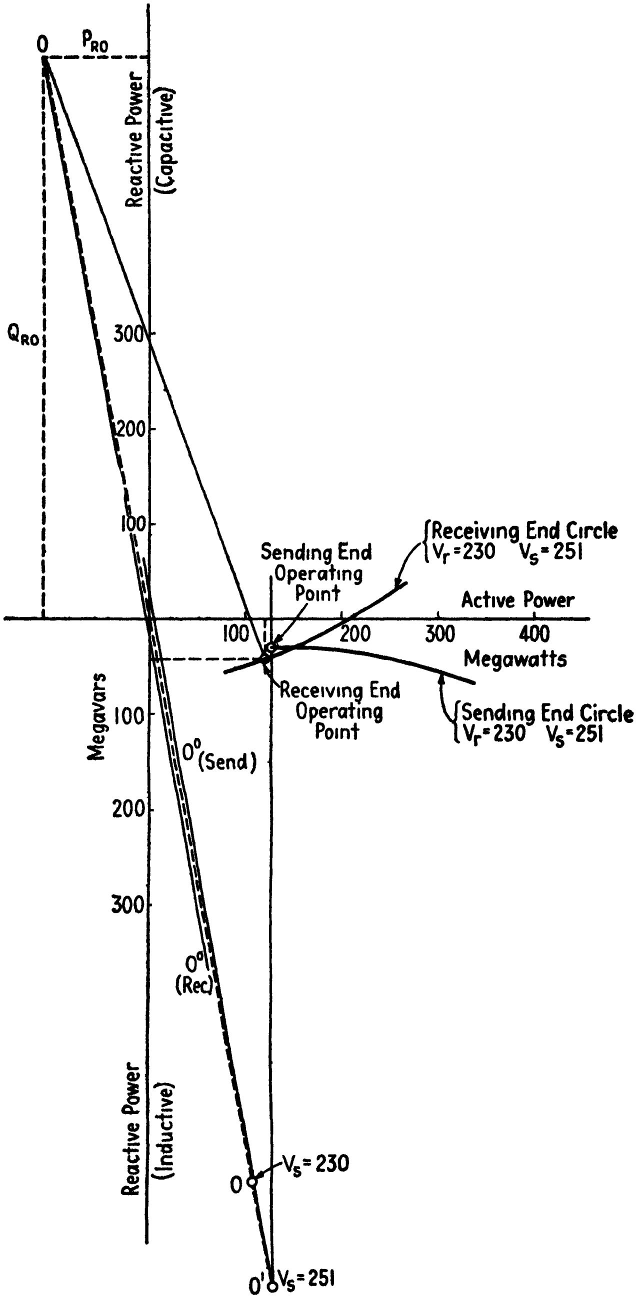
0
P_{RO}
Q_{RO}
Reactive Power (Capacitive)
300
200
100
Sending End Operating Point
Receiving End Circle $V_r=230$ $V_s=251$
Active Power
100
200
300
400
Megawatts
Receiving End Operating Point
Sending End Circle $V_r=230$ $V_s=251$
Megavars
100
200
300
0° (Send)
0° (Rec)
Reactive Power (Inductive)
0
$V_s=230$
0'
$V_s=251$

FIG. 11-9. Change of center of circle diagram corresponding to change of V_S.

line and 0.0475 inches normal to it, and drawing the zero-degree line through the point thus located.

The operating point is located at $P_R = 120$ megawatts, $Q_R = 40$ megavars inductive. The radius of a circle through the operating point (the distance from O) is measured and is found to be (on the coordinate scale) 673.

It is now convenient to compute the radius of a circle on which operating points would fall if both V_S and V_R were 230 kilovolts. Equation 11-69 is used, and gives a radius of 616. It therefore follows that since the radius to the actual operating point is 673, as compared to 616 for an assumed voltage of 230 kilovolts, the actual voltage is $(230)(673/616) = 251$ kilovolts. The angle from the zero-degree line to the operating point is measured as 10.0 degrees.

Next, the sending-end diagram is to be drawn. We shall use the same axes in Fig. 11-9. The point O for the sending-end diagram, near the bottom of the figure, could be located by using Eqs. 11-71 and 11-72, but since the transmission line under consideration is entirely symmetrical, the general circuit constant D equals A, and the sending-end O is symmetrical with the receiving-end O if $V_S = V_R$. Let us therefore locate the point O for the sending-end diagram on the same dash line that passes through the receiving-end O and the origin; we measure the distance from the origin to the receiving-end O as 600 (on the coordinate scale), so the same distance is measured to sending-end O. However, it is known from the results obtained above that V_S is actually 251 kilovolts, not 230. Hence (by Eqs. 11-71 and 11-72) the actual center of the sending-end circle, although it lies on the dash line as drawn, is not at a distance of 600 from the origin, but at a distance of $(600)(251/230)^2$ or 715. Consequently, a distance of 715 is measured along the dash line, and the point O' is located.

A sending-end zero-degree angle line is next drawn, making an angle δ (which equals α) with the dash line. Another radial line is drawn 10.0 degrees from this reference line, an arc is struck with radius 673 (same radius as the receiving-end circle), and the point of intersection of the arc and the radial line is the sending-end operating point.

The coordinates of this sending-end operating point are read as $P_S = 125$ megawatts and $Q_S = 30$ megavars. This completes the solution of the problem.

It is evident that the horizontal distance between the sending-end operating point and the receiving-end operating point represents the loss of power in the line; it is 5 megawatts. The vertical distance between the operating points indicates that the line delivers 40 megavars of inductive reactive power and takes in only 30; hence the line itself (because of its capacitance) is acting as a source of 10 megavars of inductive reactive power.

This example illustrates the use of what is sometimes called "ratioing" in the construction of new circles. The general principles are as follows:[15] After one sending-end circle and one receiving-end circle have been drawn for a given network, all further construction of the diagrams can be done by simple ratios.

1. The centers of receiving circles are all on a straight line through the origin, the distance from the origin being proportional to V_R^2.
2. The centers of sending circles are all on a straight line through the origin, the distance from the origin being proportional to V_S^2.
3. The radii of both sending and receiving-end circles are proportional to the product $V_S V_R$.
4. The line for zero angle must be redrawn when a new center point is located, but its direction does not change and all zero-degree lines are parallel.

11. Transmission-line Charts. A diagram can be drawn for a transmission line with a family of circles to represent many different voltages, and a family of radial lines to represent different angles. A diagram so prepared can be used as a chart for reference, not in one problem alone, but in any problem relating to that transmission line.

If, for example, a number of other circles are drawn in Fig. 11-8 for different values of V_S, keeping V_R the same for all, such a chart results. It has the appearance of Fig. 11-10. However, a chart made in this manner would be applicable at only one value of receiving-end voltage (as is Fig. 11-8*a*), although it would have circles for many values of sending-end voltage. To use a chart at various receiving-end voltages, the coordinates of the center of the circles must have values of P_R and Q_R that depend on the value of V_R. There are two ways that this can be accomplished: either the center point can be moved when a new value of V_R is to be used, or the coordinate scales can be changed when V_R is changed so that the same center point applies for all values of V_R.

The latter expedient has been adopted in Fig. 11-10, to produce what is known as the "modified Evans and Sels" chart.[16] The quantities read from the modified chart are not P and Q directly, but P and Q divided by a factor that depends on the square of the actual voltage. Thus the scale of the whole chart is changed when V_R is changed, and this change of scale accomplishes the same purpose as the process of *ratioing*, automatically

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[15] Adapted from the Westinghouse "Reference Book," p. 338 (Ref. 18).

[16] Much of the basic work on circle charts is owing to R. D. Evans and H. K. Sels, Circle Diagrams for Transmission Systems, *Elec. J.*, December, 1921, p. 530. "Modified" scales were soon thereafter proposed by several authors. The chart herein described is only one of several that are applicable at all voltages. This one has the advantage of being direct-reading at rated voltage (see Refs. 5, 16, and 38).

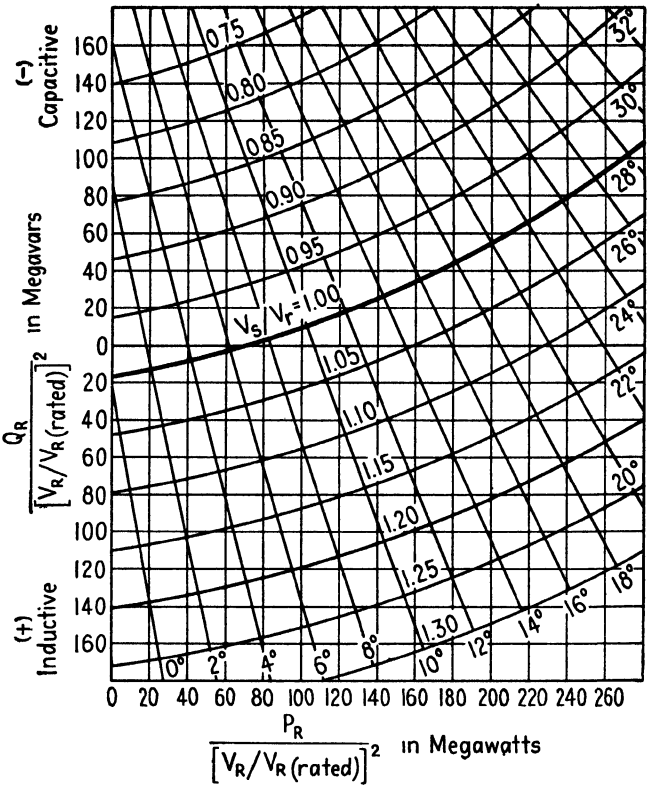

FIG. 11-10. Receiving-end chart. P_R is active power delivered to the load. Q_R is reactive power delivered to the load, and if positive (to an inductive load) is measured downward.

altering both the centers and the radii of the circles when a new value of V_R is used.

To show that the modified chart accomplishes this purpose, divide each term of Eq. 11-66 by the ratio $(V_R/V_{R(\text{rated})})^4$, where V_R has its usual meaning of receiving-end voltage, and $V_{R(\text{rated})}$ is the rated value of receiving-end voltage. The result is

$$\left[\frac{P_R}{(V_R/V_{R(\text{rated})})^2} + \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}(V_{R(\text{rated})})^2\right]^2 + \left[\frac{Q_R}{(V_R/V_{R(\text{rated})})^2} - \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}(V_{R(\text{rated})})^2\right]^2 = \frac{(V_{R(\text{rated})})^4}{b_1^2 + b_2^2}\left[\frac{V_S}{V_R}\right]^2 \qquad (11\text{-}74)$$

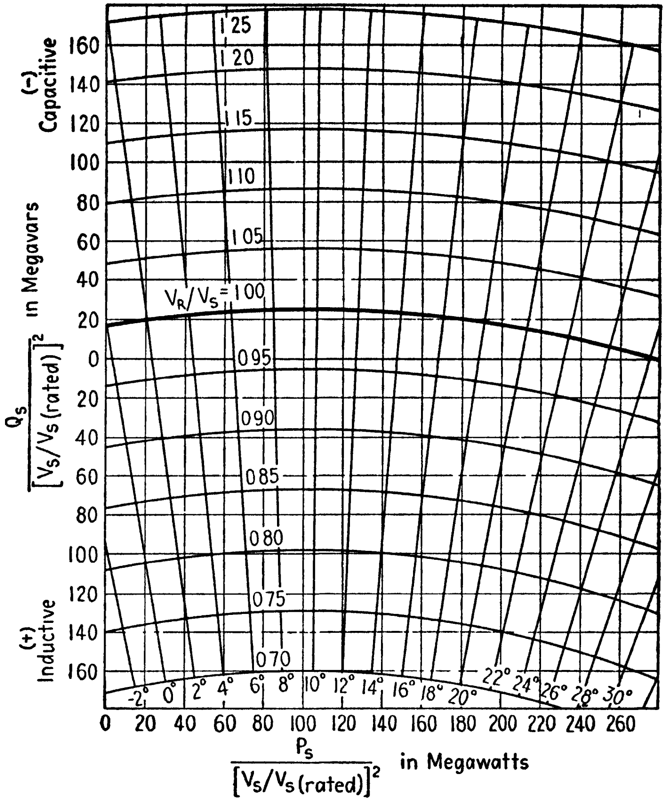

FIG. 11-11. Sending-end chart. P_S is active power delivered to the line. Q_S is reactive power delivered to the line, and if positive (as to an inductive reactance) is measured downward.

This is recognized as the equation of a circle if the variables are those shown in Fig. 11-10, the center of the circle being at the point:

$$\text{Abscissa} = -\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}(V_{R(\text{rated})})^2 \tag{11-75}$$

$$\text{Ordinate} = +\frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}(V_{R(\text{rated})})^2 \tag{11-76}$$

and the radius being

$$\text{Radius} = \frac{(V_{R(\text{rated})})^2}{\sqrt{b_1^2 + b_2^2}}\left(\frac{V_S}{V_R}\right) \tag{11-77}$$

(The symbols are defined below Eq. 11-69.)

Equation 11-74 is the equation of any one of the circles of the *receiving-end chart* of Fig. 11-10. The coordinates of the center do not change with changes of operating voltage; they are determined by the rated receiving voltage which is an arbitrary and fixed value. Hence all circles have the same center point, regardless of the actual operating values of voltage at either end of the line. The radius of each individual circle depends on the ratio of voltages, V_S/V_R. Thus, each circle is the locus of operating points corresponding to constant V_S/V_R.

If V_R equals $V_{R(\text{rated})}$, Eqs. 11-75 to 11-77 are the same as 11-67 to 11-69. At rated receiving voltage, therefore, the modified chart is the same as the prototype chart, and it is direct-reading in megawatts and megavars on the horizontal and vertical axes. At other receiving voltages, values read from the coordinates must be multiplied by $(V_R/V_{R(\text{rated})})^2$ to obtain actual values of active and reactive power.

It is evident that a similar modification of Eq. 11-70 will yield a circle chart for sending-end P and Q, as in Fig. 11-11, universal as regards terminal voltages. The center of the circles on the *sending-end chart* has the coordinates

$$\text{Abscissa} = \frac{d_1b_1 + d_2b_2}{b_1^2 + b_2^2}(V_{S(\text{rated})})^2 \tag{11-78}$$

$$\text{Ordinate} = -\frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}(V_{S(\text{rated})})^2 \tag{11-79}$$

and the radius of each circle is

$$\text{Radius} = \frac{(V_{S(\text{rated})})^2}{\sqrt{b_1^2 + b_2^2}}\left(\frac{V_R}{V_S}\right) \tag{11-80}$$

Angle lines are located on the modified charts just as on the prototype charts of Fig. 11-8.

Such a pair of charts may be drawn for a transmission line, one for sending and one for receiving, and thereafter most problems relating to operation of the line can be answered by reference to the charts. A numerical example will illustrate the method of using the charts.

Example. The charts for a transmission line are shown in Figs. 11-10 and 11-11. These are for the Oroville-to-Bellota line, the dimensions of which are given in the example on page 288. The constants a_1, a_2, b_1, and b_2 were computed in the previous example from the line dimensions.

For preparing the modified charts, the nominal or rated voltage is taken to be 230 kilovolts at each end of the line. Using this value, and the constants previously computed, the charts are drawn.

As an example of the use of the charts, consider that the load on the line is 120 megawatts power and 40 megavars reactive (inductive load). Receiving voltage is 230 kilovolts. The operating point is found on the receiving chart; $P_R = 120$ and $Q_R = 40$ are read directly on the axes, as $V_R/V_{R(\text{rated})} = 1$. From the chart, $V_S/V_R = 1.090$, and the angle between V_S and V_R is 10.0 degrees.

Compute $V_R/V_S = 0.918$. Enter the sending chart with this voltage ratio and the angle 10.0 degrees. Read on the axes of the chart the values 105 megawatts and 25 megavars (inductive). These values must be corrected. Compute $V_S = (1.090)(230) = 251$ kilovolts, and find the ratio $(V_S/V_{S(\text{rated})})^2 = 1.188$. Multiply the chart readings by this correction factor: $P_S = (105)(1.188) = 125$ megawatts and $Q_S = (25)(1.188) = 30$ megavars (inductive).

The fundamental quantities are now known. From them we may find power loss in the line as $125 - 120 = 5$ megawatts. Line efficiency $= 120/125 = 0.96$. Production of reactive power in the line $= 40 - 30 = 10$ megavars.

It is evident that this is only one of the simpler of a great variety of problems that can be solved by use of the charts.

12. Line Loss. Loss, in the above examples, is computed as the small difference of two relatively large quantities. The accuracy, particularly when the larger quantities are obtained by graphic methods, is often unsatisfactory. Power loss must usually be known with good accuracy because of its direct economic importance in transmission line design.

Loss values obtained from charts in the manner illustrated are helpful for exploratory work, and, when careful studies require more exact values, it is usual to resort to computation. Various formulas can be used (see Refs. 5, 18, and 38). The following is one of the most convenient:

$$\begin{aligned}\text{Loss} = {} & (a_1c_1 + a_2c_2)V_R^2 + 2(a_2d_2 + b_1c_1)P_R \\ & + 2(a_1d_2 - b_1c_2)Q_R + (b_1d_1 + b_2d_2)\frac{P_R^2 + Q_R^2}{V_R^2} \qquad (11\text{-}81)\end{aligned}$$

This expression[17] gives three-phase loss in megawatts in terms of the general circuit constants of the line and

V_R, receiving-end line-to-line voltage (kilovolts)
P_R, received three-phase active power (megawatts)

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[17] The derivation is somewhat lengthy but basically simple. It is given by Dahl and Woodruff in Refs. 5 and 38. The principal steps are as follows: Loss $= \operatorname{Re}\{V_s\bar{I}_s - V_r\bar{I}_r\}$. Eliminate V_s and I_s by substituting V_r, I_r, and the constants $ABCD$. Expand $ABCD$ and I_r into real and imaginary components, retaining only the real component of the quantity in brackets. Expand $AD - BC = 1$ in the real and imaginary components of $ABCD$, using the result to simplify the expression for loss. Express current in terms of active and reactive power and voltage. Change from volts, watts, and vars per phase to kilovolts line to line, and three-phase megawatts and megavars. Equation 11-81 results.

Q_R, received three-phase reactive power (megavars, positive if to an inductive load)

When this formula is used in the numerical example of the previous section, the computed loss is found to be 4.52 megawatts, compared with the graphically determined value of 5.

Note that the last of the four terms in the expression for loss is approximately the resistance of the line times the square of the current in the line, and this term accounts for most of the loss when the load is heavy.

It is possible to rewrite Eq. 11-81 in a form that is recognizable as describing a family of circles on a transmission-line chart. The circles are constant-loss loci. If these circles are drawn, power loss in the line can be read from the chart (see Refs. 5, 8, 14, 16, and 38). However, the computation required in finding the radii of the circles is considerable. It seems that most engineers engaged in transmission-line calculations prefer to find loss by subtraction, using values of P_R and P_S obtained from charts if the approximate result that can be obtained by this method is satisfactory, and to employ numerical computation rather than a graphic solution when precision is needed.

In conclusion, other loci that can appear on a transmission-line diagram may be mentioned. *Lines of constant power factor* comprise a family of straight lines radiating from the origin of coordinates, the power factor represented by each being the cosine of its angle with the horizontal axis. A few such lines, corresponding to known or assumed load power factors, are often useful on the receiving-end chart.

Circles of constant efficiency of transmission can be drawn; so, also, can circles of constant active power at the distant end of the line, and circles of constant reactive power at the distant end. A complete treatment of circle diagrams is given by Lewis (Ref. 16).

There are two major alternatives in the use of transmission-line diagrams. One method is to draw complete charts as illustrated in Figs. 11-10 and 11-11. The other is to draw axes and to locate centers, but not to draw families of circles and angle lines; then, when any specific problem is to be solved, the pertinent circles and angle lines are drawn, using specific values that relate to that particular problem. The latter method has the advantage of greater precision, for no interpolation is necessary as on the charts; radii, in particular, can be measured with greater precision than is possible from reading an ordinary chart.

When a pair of circle diagrams or charts, of whatever type the engineer prefers, have been prepared, they will be filed for further use when other problems relating to that particular line arise. It is helpful for the engineer in charge of a transmission system to keep a complete set of circle diagrams relating to each major link of his network.

13. Corona Loss. High-voltage power lines are subject to a type of energy loss that does not occur on lines operating at low voltage. The high electric field strength in the region immediately around the high-voltage conductors causes ionization of air in that region. Electrons are accelerated by the force of the electric field and, if the voltage is sufficiently high, electrons will gain enough energy to enable them to ionize air molecules by collision. If the electric field strength exceeds a critical value, ionization becomes cumulative, and appreciable loss of power from the high-voltage line results.[18] The luminosity that accompanies profuse ionization is called *corona*; it is readily visible at night about the surface of overstressed high-voltage conductors. Under ideal conditions, it appears as a uniform glow, but more commonly it is in the form of a multitude of *brush discharges* from minor roughnesses and irregularities of the conductor surface.

Good engineering practice demands that corona loss be avoided. It is not practical for the loss to be zero on a high-voltage power line, but it should be kept below some such nominal value as 1 or 2 kilowatts per mile of three-phase line in fair weather.

Corona loss is not readily taken into account in the mathematics of transmission lines. Since operating voltages are commonly held below the voltage of profuse corona formation, it is not often necessary to do so.[19] From the practical point of view, it is most important to know the voltage at which corona loss becomes excessive so that this voltage may not be exceeded.

Corona loss is affected by many factors, some having to do with the conductor surface, and some with atmospheric conditions. A complete theoretical treatment is impractical, but two empirical methods of computing corona loss are available and both give dependable results in the range of conditions to which they are adapted; one is owing to Peterson, the other to Carroll and Rockwell.[20] Both methods of computation take into account air temperature and pressure, frequency, spacing between conductors, size of conductors, and stranding and surface condition of conductors. Since both methods require careful and thoughtful application for good accuracy, no summary will be attempted. The reader who finds it necessary to compute corona loss should refer to the original papers.

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[18] Ionization by collision or impact is the basic process of corona formation about a high-voltage conductor. The theory of ionization by collision, first proposed by J. S. Townsend in 1904 (Townsend, "Electricity in Gases," Oxford University Press, New York, 1915) has been extensively discussed in the literature of physics and electrical engineering. Much of the original investigation of corona was done by H. J. Ryan (see *Trans. AIEE*, 1911, pp. 1–130) and C. P. Steinmetz (see F. W. Peek, "Dielectric Phenomena in High Voltage Engineering," 3d ed., McGraw-Hill Book Company, Inc., New York, 1929). More recent work has been done by J. S. Carroll (see footnote 20) and others. Corona in ordinary air is a rather complicated combination of impact ionization, streamer formation, and space-charge shielding; a brief description is given in discussion by H. H. Skilling, *Trans. AIEE*, March, 1932, p. 79.

[19] The voltage of a traveling wave caused by lightning, however, may greatly exceed the corona-forming voltage of the line on which it travels. If so, corona loss causes rapid attenuation of the wave, and a curious change of wave form. See H. H. Skilling and P. de K. Dykes, Distortion of Traveling Waves by Corona, *Trans. AIEE*, 1937, p. 850.

The following excerpts from the Carroll and Rockwell paper give an idea of the voltage at which corona becomes excessive. For anything more than a general indication of corona-forming voltage, the corrections described in the original paper must be made.

Carroll and Rockwell show that loss from an ordinary high-voltage power line may conveniently be accounted for by finding the voltage at which the loss per mile of three-phase line becomes 4 kilowatts. For stranded copper or aluminum cable, the surface being of normal roughness and well weathered, suspended with 22-foot flat spacing, normal atmospheric temperature and pressure at sea level, at a frequency of 60 cycles per second, the line-to-line voltage at which three-phase corona loss per mile is 4 kilowatts is

$$V = 167D + 107 \text{ kilovolts rms sine-wave equivalent} \qquad (11\text{-}82)$$

D is the outside diameter of the conductor in inches. This expression is applicable for values of D between 0.7 and 2.1. For segmental copper (Type HH) conductor, with D between 1.0 and 1.5 inches, use the same formula and add 40 kilovolts to V.

Having determined the voltage at which loss is 4 kilowatts per mile, the voltage corresponding to another value of loss per mile is judged by reference to typical experimental curves. For instance, the loss is 2 kilowatts per mile (a commonly acceptable fair-weather design value) at a voltage that is lower than the 4-kilowatt-loss value by something between 15 and 30 kilovolts. The loss is 1 kilowatt per mile at a voltage that is 40 to 60 kilovolts below the 4-kilovolt-loss voltage. Correction is then made for air density and other factors. Air density will probably not change the critical voltage by more than a few per cent unless the line is in operation at high elevation above sea level.

It must be repeated that this brief statement permits only a crude determination of the voltage at which corona is to be expected. If, by subtracting 60 kilovolts from the value computed from Eq. 11-82, one obtains a corona-forming voltage that is still 25 per cent or so above the operating voltage of a line, and the line is normal in design and location, it is reasonable to expect that it will have negligible corona loss. If, on the other hand, the operating voltage approaches closely the corona-forming voltage, full use should be made of the original papers cited.

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[20] W. S. Peterson, *Trans. AIEE*, March, 1933, p. 62; J. S. Carroll and M. M. Rockwell, *Trans. AIEE*, 1937, p. 558. Both are based primarily on experimental work at Stanford University by Carroll and associates: see references in the latter paper.

For lines of exceptional design, there is no substitute for actual measurement under conditions approximating those of anticipated operation.

PROBLEMS

11-1. Switchboard instruments in a generating station with 39,000 kilovolt-amperes installed capacity indicate 25,300 kilowatts, 13,000 kilovars, 10.5 kilovolts. Power is three-phase and voltage is line to line on low-voltage bus. Find (*a*) station power factor, (*b*) kilovolt-ampere output, and (*c*) bus current.

11-2. A three-phase generating station with 54,000 kilovolt-amperes installed capacity is developing full-rated kilovolt-amperes at 0.85 power factor (lagging). The bus voltage is 12,000 volts, line to line. Find (*a*) active power and reactive power output and (*b*) bus current.

11-3. Is reactive power produced or consumed (using the convention adopted in this book) by the following:

a. Synchronous motor, overexcited.
b. Transformer, magnetizing current.
c. Generator, underexcited.
d. Series capacitor.
e. Synchronous condenser, underexcited.
f. Induction motor, lightly loaded.

11-4. It is sometimes stated that "with reference to line voltage, reactive power flows down hill." Restate this quotation in more precise words, and show that it is true under ordinary conditions of power transmission.

11-5. A 220-kilovolt, three-phase, 60-cycle power transmission circuit (Conowingo-Plymouth Meeting) is 58 miles long. The three conductors are each 795,000-circular-mil ACSR with 30 aluminum and 19 steel strands; they are suspended with 25.5-foot separation in a horizontal plane. Assume the temperature is 25°C. Find Z, Y, and ZY (Eq. 11-14) of the line.

11-6. A transmission line (Oroville-Bellota) is described in the example on page 288. Find Z, Y, and ZY (Eq. 11-14) of this line.

11-7. The line of Prob. 11-5 is to be represented on a network analyzer by an equivalent Π section. Determine values of R, X, and Y to use on the analyzer.

11-8. The line of Prob. 11-6 is to be represented on a network calculator by an equivalent Π section. Determine values of R, X, and Y to use on the calculating board.

11-9. Repeat Prob. 11-7 for an equivalent T section.

11-10. Repeat Prob. 11-8 for an equivalent T section.

11-11. From the results of Prob. 11-5, find the general circuit constants of the Conowingo-Plymouth Meeting line.

11-12. From the results of Prob. 11-6, find the general circuit constants of the Oroville-Bellota line.

11-13. Find the general circuit constants of the Π section of Prob. 11-7 (or of the T section of Prob. 11-9), thereby obtaining the transmission-line general circuit constants indirectly.

11-14. Find the general circuit constants of the Π section of Prob. 11-8 (or the T section of Prob. 11-10), thereby obtaining the transmission-line general circuit constants indirectly.

11-15. It is stated that the error in the expressions of Fig. 11-2 for equivalent Π and T artificial lines "is approximately proportional to the fourth power of length." Show that this is true.

11-16. Derive Eq. 11-5.

11-17. Derive the formulas of Fig. 11-4 for $ABCD$ of the nonsymmetrical Π section.

11-18. Derive the formulas of Fig. 11-4 for $ABCD$ of the nonsymmetrical T section.

11-19. Show that $AD - BC = 1$ for a nonsymmetrical Π section.

11-20. Show that $AD - BC = 1$ for a nonsymmetrical T section.

11-21. Show from Eqs. 11-49 that $A_0 = D_0$ for two networks in cascade, even though each network by itself is an unsymmetrical network, provided the networks are identical and the two are so connected (with like ends together) that together they comprise a symmetrical system.

11-22. Derive $A_0B_0C_0D_0$ for two networks in parallel, and show that these reduce to Eqs. 11-50 if the two networks are identical.

11-23. Use a circle diagram to determine the following values for the transmission line of Prob. 11-6 (Oroville-Bellota). See Fig. 11-9. Draw the diagram with care on a sheet of paper at least 10 by 16 inches; coordinate paper is recommended for convenience. Do not fold the sheet, for folding impairs accuracy.

a. Received voltage is 230 kilovolts; received power 105 megawatts, unity power factor. Find sending-end voltage, P and Q, power loss, and efficiency of transmission.

b. Sending voltage and receiving voltage are both 230 kilovolts; received power 120 megawatts. Find received Q and sending P and Q.

c. Sending voltage is 230 kilovolts; sending power is 120 megawatts at unity power factor. Find receiving V, P, and Q.

d. Receiving voltage is 230 kilovolts; sending power is 120 megawatts at unity power factor. Find sending voltage and receiving P and Q.

e. What synchronous condenser capacity is required at the receiving end to maintain voltage at 230 kilovolts at both ends with no load?

f. What synchronous condenser capacity is required at the receiving end to maintain voltage at 230 kilovolts at both ends of the line; received power is 100 megawatts and receiving-end power factor may vary from 0.80 (current lagging) to 1.00?

11-24. Repeat Prob. 11-23 for the transmission circuit of Prob. 11-5 (Conowingo-Plymouth Meeting).

11-25. Use the charts of Figs. 11-10 and 11-11 to obtain the results required in

Prob. 11-23. If you have obtained the same results by the method of Prob. 11-23, compare (*a*) ease and (*b*) accuracy of the two methods.

11-26. Draft a pair of charts, similar to Figs. 11-10 and 11-11, for the Conowingo-Plymouth Meeting transmission circuit of Prob. 11-5.

11-27. Use the charts of Prob. 11-26 to obtain for the Conowingo-Plymouth Meeting circuit the results required in Prob. 11-23.

11-28. The Oroville-Bellota line is used as a numerical example in Secs. 10 and 11. For the given load, compute power loss in transmission. The computed value should have slide-rule accuracy.

11-29. Problem 11-24 requires determination by graphic means of power loss in transmission. (Part *a* of the problem.) For the same conditions, compute the power loss. The computed value should have slide-rule accuracy.

11-30. A 220-kilovolt, three-phase, 50-cycle power line in Sweden is 485 kilometers long. The conductors are copper, 25.0 millimeters outside diameter, 255 square millimeters cross section; they are suspended with 7.90-meter separation in a horizontal plane. Problems 11-5, 11-7, 11-9, 11-11, 11-13, 11-23, 11-26, and 11-27 may be applied to this Swedish line.

11-31. Is voltage on the line of Prob. 11-5 high enough to make corona loss an important factor in the design of the line?

11-32. The Hoover Dam line of Fig. 5-3 uses segmental copper (Type HH) conductor. Rated sending-end voltage is 287.5 kilovolts. Is corona loss an important factor in the design of the line?

CHAPTER 12

APPLICATION TO RADIO-FREQUENCY LINES

A. RADIO-FREQUENCY RELATIONS

1. Use of Lines at High Frequency. Lines are used at radio frequency to transmit energy, much as they are at lower frequencies, and in addition, they are used for their impedance characteristics. When frequency is high enough to make a quarter-wavelength comparable to the dimensions of laboratory apparatus, lines can be used as circuit components.

Various types of line are shown in Fig. 12-1. The coaxial line is far more commonly used than the parallel-wire line at high frequencies because it does not radiate energy into space, the electric and magnetic fields being enclosed within the outer tubular conductor. Also, the coaxial line is shielded by its outer conductor from interference from nearby circuits. A double coaxial line can be used if the circuit must be balanced to ground. Another balanced line is a two-wire line enclosed within a shielding copper tube.

The *upper limit of frequency* for which a coaxial line is useful is the frequency at which a wavelength is equal to the average circumference of the line. Lines will propagate waves of any frequency, however high, but if the wavelength of the signal on the line is less than $(a + b)\pi$, a and b being radii of the inner and outer conductors, respectively, a coaxial line is capable of propagating other modes than the usual transmission-line mode (see Sec. 11, Chap. 13) and this is detrimental to simple and efficient operation. A similar difficulty is found on parallel-wire lines if the spacing between the wires is greater than $\frac{1}{2}$ wavelength of the signal frequency, but the practical upper limit of frequency for open parallel-wire lines is much lower because of radiation loss.

2. Approximations at High Frequency. Obviously, line loss cannot be neglected if it is the determining factor in a problem. The current in a line at a point $\frac{1}{4}$ wavelength from a short circuit, for instance, is primarily dependent on line loss, for in an ideal line it would be zero. Similarly, the input impedance to a quarter-wavelength short-circuited line is determined by line loss. The resonant properties of an open or short-circuited line are dependent on loss. In general, loss is the determining factor at nodal points on resonant lines where the electrical quantities of interest tend toward zero or infinity.

In radio-frequency problems not involving quantities at nodal points, loss is commonly neglected. This is not because loss is less at radio frequency; indeed, loss is greater than in the same line at lower frequency. But radio-frequency lines are usually short; their lengths may be something between a few centimeters and a few hundred meters, compared to telephone or power lines of many miles. Energy loss is usually small in the length of a line.

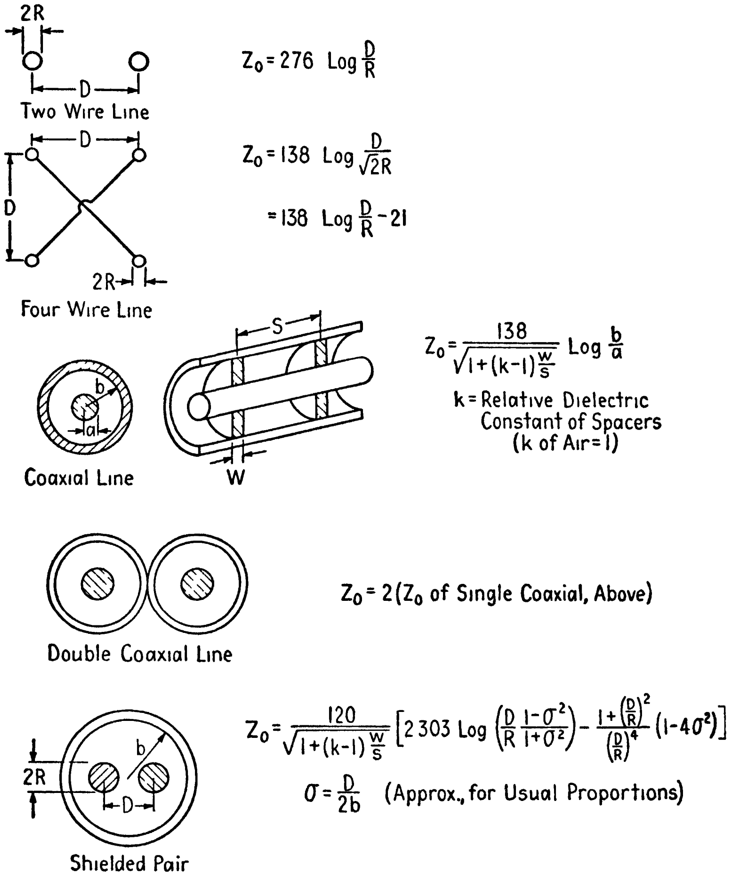
2R
D
Two Wire Line
$Z_0 = 276 \text{ Log } \frac{D}{R}$
D
D
2R
Four Wire Line
$Z_0 = 138 \text{ Log } \frac{D}{\sqrt{2}R}$
$= 138 \text{ Log } \frac{D}{R} - 21$
b
a
Coaxial Line
S
W
$Z_0 = \frac{138}{\sqrt{1+(k-1)\frac{w}{s}}} \text{ Log } \frac{b}{a}$
k = Relative Dielectric Constant of Spacers (k of Air = 1)
Double Coaxial Line
$Z_0 = 2(Z_0$ of Single Coaxial, Above)
2R
b
D
Shielded Pair
$Z_0 = \frac{120}{\sqrt{1+(k-1)\frac{w}{s}}} \left[2.303 \text{ Log } \left(\frac{D}{R}\frac{1-\sigma^2}{1+\sigma^2}\right) - \frac{1+\left(\frac{D}{R}\right)^2}{\left(\frac{D}{R}\right)^4}(1-4\sigma^2)\right]$
$\sigma = \frac{D}{2b}$ (Approx., for Usual Proportions)

FIG. 12-1. Common radio-frequency lines.

Equations neglecting line loss can be taken from Chap. 3, as follows.

As *approximations*

$$Z_0 = \sqrt{\frac{l}{c}} \tag{3-3}$$

$$\beta = B = \omega\sqrt{lc} \tag{3-5}$$

$$\gamma = j\beta = j\omega\sqrt{lc} \tag{3-4}$$

$$\text{Velocity} = \frac{\omega}{\beta} = \frac{1}{\sqrt{lc}} \tag{3-8}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{lc}} \tag{3-9}$$

$$V = V_1 e^{j\beta x} + V_2 e^{-j\beta x} \tag{3-6}$$

$$I = I_1 e^{j\beta x} + I_2 e^{-j\beta x} \tag{3-7}$$

$$V = V_r\left(\cos\beta x + j\frac{Z_0}{Zr}\sin\beta x\right) \tag{3-14}$$

$$I = I_r\left(\cos\beta x + j\frac{Z_r}{Z_0}\sin\beta x\right) \tag{3-16}$$

$$\frac{Z}{Z_0} = \frac{(Z_r/Z_0)\cos\beta x + j\sin\beta x}{\cos\beta x + j(Z_r/Z_0)\sin\beta x} \tag{3-36}$$

When it is further specified that frequency is high, and that therefore the factor K_φ in the inductance formulas of Table I is practically zero, we may write

Per meter of two-wire line	*Per meter of coaxial line*	
$l = 0.921 \log\frac{b}{a} \times 10^{-6}$	$l = 0.461 \log\frac{b}{a} \times 10^{-6}$	(12-1)
$c = \frac{12.08 \times 10^{-12}}{\log(b/a)}$	$c = \frac{24.16k \times 10^{-12}}{\log(b/a)}$	(12-2)

Using these expressions for l and c in Eq. 3-3 gives
For two-wire line in air:

$$Z_0 = 276 \log\frac{b}{a} \tag{12-3}$$

For coaxial line:

$$Z_0 = \frac{138}{\sqrt{k}} \log\frac{b}{a} \tag{12-4}$$

(As before, for the two-wire line, a is radius of each wire and b is spacing center to center; for the coaxial line, a is outer radius of the inner conductor and b is inner radius of the outer conductor. Relative dielectric constant is k. Logarithms are to the base 10.) Values of Z_0 for certain other types of lines are given in Fig. 12-1.

When Eqs. 12-1 and 12-2 are used for l and c in Eq. 3-8, it is found that velocity is independent of all dimensions of a line in air. On a coaxial line with solid insulation, velocity is independent of everything except the dielectric constant of the insulating material:

$$\text{Velocity} = \frac{2.998 \times 10^8}{\sqrt{k}} \quad \text{meters per second} \tag{12-5}$$

where k is the relative dielectric constant of the insulation.

Velocity of high-frequency waves on a transmission line is the same as the velocity of electromagnetic waves radiated freely. The transmission line is fundamentally a wave guide; it directs the travel of waves that exist in the space between the conductors of the transmission line. If this space is filled with air, the velocity is the velocity of waves in air; for all practical purposes 3×10^8 meters per second. If the space is filled with a dielectric material, the velocity is that of waves in the dielectric material.

Using this value of velocity in determining the wavelength,

$$\lambda = \frac{3 \times 10^8}{f\sqrt{k}} \quad \text{meters} \tag{12-6}$$

Similarly, β may be written in terms of velocity:

$$\beta = \frac{2\pi f\sqrt{k}}{3 \times 10^8} \quad \text{per meter} \tag{12-7}$$

3. Radio-frequency Resistance. When line loss is neglected, $\alpha = 0$. If line loss cannot be neglected, Eq. 5-16 gives an approximate value for α, good at high frequency:

$$\alpha = \frac{lg + rc}{2\sqrt{lc}} = \frac{gZ_0}{2} + \frac{r}{2Z_0} \tag{12-8}$$

If limited to a *line in air*, with no dielectric losses and hence zero g, α can be further simplified to

$$\alpha = \frac{r}{2Z_0} \tag{12-9}$$

If, on the contrary, dielectric loss is much greater than resistance loss, as may occur with *solid-dielectric* coaxial lines at centimeter wavelengths, it is approximately true that

$$\alpha = \frac{gZ_0}{2} \tag{12-10}$$

To find a numerical value for α from Eq. 12-8 or 12-9, it is necessary to know the resistance per unit length. This is quite different from the low-frequency resistance of the same conductors because of skin effect. High-frequency resistance is found by using the hypothetical *effective depth of penetration* or "skin depth" from Eq. 7-35. For any *nonmagnetic* material

$$\delta = 503\sqrt{\frac{\rho}{f}} \qquad \text{meters} \tag{12-11}$$

where f is frequency (cycles per second)
ρ is resistivity of the conductor material (meter-ohms).

This equation is valid for all ordinary materials except iron and its alloys, and these are never used for the conductors of radio-frequency lines. Resistivities of silver, copper, and ordinary brass at room temperature (20°C) are

	Meter-ohms
Silver	1.63×10^{-8}
Copper	1.72×10^{-8}
Brass (approx)	7.0×10^{-8}

Table 7-2 gives values for other metals.

Resistance is found by multiplying the equivalent depth of penetration by the perimeter of the conductor, to obtain the cross section in which current is assumed to flow. Note that this method is valid for a conductor of any shape provided its thickness is at least 10δ and it has no sharp corners. Dividing resistivity of the material by this cross section in which current is assumed to flow, the resistance of unit length of the conductor is obtained.

The high-frequency resistance per unit length of both of a *pair of round wires* of radius a is therefore

$$r = 2\frac{\rho}{2\pi a\delta} \tag{12-12}$$

Using values of ρ as given above in meter-ohms, and δ in meters from Eq. 12-11, Eq. 12-12 will give resistance in ohms in the unit of length used to measure the radius a. Thus, if a is measured in meters, r is in ohms per meter.

Combining the above equations, the high-frequency resistance of a pair of wires of nonmagnetic material is

$$r = \frac{\sqrt{\rho}}{503\pi} \cdot \frac{\sqrt{f}}{a} \tag{12-13}$$

It is important that the high-frequency resistance is proportional to the square root of the frequency.

To find the resistance of a *coaxial line,* the resistance of the current-carrying inner surface of the outer tube is added to the resistance of the one wire:

$$r = \frac{\rho}{2\pi a \delta} + \frac{\rho}{2\pi b \delta} = \frac{\rho}{2\pi \delta}\left(\frac{1}{a} + \frac{1}{b}\right) \tag{12-14}$$

For any nonmagnetic material, the high-frequency resistance of a coaxial line is thus

$$r = \frac{\sqrt{\rho}\sqrt{f}}{1006\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \tag{12-15}$$

The resistance of a coaxial line also is proportional to the square root of frequency.

Finally, the resistivity of the material can be combined with the other constants involved, and Eqs. 12-13 and 12-15 can be written to give the high-frequency resistance of lines of nonmagnetic material as:

For two-wire line:

$$r = 2K\frac{\sqrt{f}}{a} \quad \text{ohms per unit length} \tag{12-16}$$

For coaxial line:

$$r = K\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{ohms per unit length} \tag{12-17}$$

where f is frequency (cycles per second)
a is the radius of the wire
b is the inner radius of the tube
$K = 41.6 \times 10^{-9}$ for copper
$= 40.2 \times 10^{-9}$ for silver
$= 83 \times 10^{-9}$ for brass (approx)

For other metals, K is proportional to the square root of resistivity. Note that a and b must be measured in the same units of length, and the result is resistance in this unit of length of transmission line. Thus, if a and b are in meters, r is in ohms per meter.

Proximity Effect. Resistance of a two-wire line, determined from Eq.

12-16, is increased by proximity effect if the spacing between wires is not large compared to the wire size. Resistance computed from the equation is to be multiplied by the appropriate factor from Table 12-1. Since b/a for two-wire lines is usually greater than 10, and is practically always greater than 5 (for which the clear space between wires is only $1\frac{1}{2}$ diameters), proximity effect is usually negligible. The error that results from neglecting proximity effect may well be less than other probable errors in determining resistance.

TABLE 12-1

PROXIMITY FACTOR IN PARALLEL WIRES OR TUBES AT VERY HIGH FREQUENCY

$\frac{\text{Spacing}}{\text{Radius}} = \frac{b}{a}$	*Proximity Factor*
5*s**	1.07
5*op*†	1.09
10	1.02
15	1.01
20	1.01

* *s* indicates current in same direction in the two wires.
† *op* indicates current in opposite directions in the wires.

Furthermore, if b/a is so small that proximity effect is significant, there is another complication in the use of transmission-line equations. The approximate value for capacitance of two-wire lines given in Eq. 12-2 must be examined critically, for it is derived with the assumption that b/a is large. The more accurate formula of Eq. 6-74a may be required.

Proximity effect is similar to skin effect. If two conductors, parallel and close together, are carrying current in the same direction, there is a tendency for the current to be more concentrated in the more distant parts of the conductors, for inductance of the outer paths is less. If, on the other hand, current is in opposite directions in two parallel conductors it tends to concentrate at the nearer surfaces,[1] as in Fig. 12-2. The resulting dissymmetry of current density increases loss in the conductor, and hence increases the effective resistance. It is obvious that proximity effect is a function of frequency, vanishing at low frequency, and also that it depends on conductor material; however, simplifying assumptions are possible if the frequency is so high that "skin depth" δ is small compared to the size of the wire, and Table 12-1 makes this assumption.

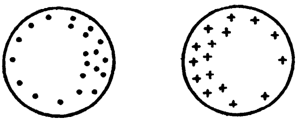

FIG. 12-2. Proximity effect.

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[1] See also the discussion of Unequal Division of Current in four wires, Chap. 6, Sec. 6.

Coaxial lines do not have any similar dissymmetry of current distribution, and hence have no proximity effect. It may be better to say that skin effect and proximity effect are indistinguishable in coaxial lines. Current following the path of least inductance tends to concentrate near the outer surface of the inner conductor and the inner surface of the outer conductor, and this whole effect is called skin effect in a coaxial line.

4. Dielectric Loss. All solid and liquid dielectric materials waste appreciable amounts of power in high-frequency electric fields if the frequency corresponds to wavelengths in the range of a few centimeters. See the discussion of insulation loss in Sec. 10, Chap. 7.

Dielectric power factors of various materials have been measured, and the experimental results are tabulated in reference books.[2] The power factor is a property of the material, as explained in Chap. 7. It is essentially independent of frequency. Nevertheless, since power factor is found to vary somewhat if the frequency is changed by several powers of 10, it is best to employ in computation an experimental value that is known to apply in the frequency range of the computation.

Equation 7-72 relates the power factor of a dielectric material to the equivalent value of g that should be used to represent that power factor:

$$g = \omega c \times (\mathrm{PF}) \tag{12-18}$$

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[2] Recent data on three high-quality dielectric materials are as follows:

Frequency	10^3	10^6	10^8	3×10^9	2.5×10^{10}
Polyethylene DE-3401:					
k*....................	2 26	2.26	2 26	2.26	2.26
PF†..................	<0 0002	<0.0002	0 0002	0.00031	0.0006
Polystyrene:					
k*....................	2.56	2.56	2.55	2 55	2.54
PF†..................	<0.00005	0.00007	<0.0001	0.00033	0.0012
Teflon:					
k*....................	2.1	2.1	2.1	2.1	2.08
PF†...	<0.0003	<0 0002	<0.0002	0.00015	0.0006

*k is relative dielectric constant.

†PF is dissipation factor (equal to power factor for these low values).

These values are abstracted from extensive tables in "Reference Data for Radio Engineers," H. J. Reich, editor, Federal Telephone and Radio Corporation, New York, 1949. Tables are also given in F. E. Terman, "Radio Engineers' Handbook," McGraw-Hill Book Company, Inc., New York, 1943, and T. Moreno, "Microwave Transmission Design Data," McGraw-Hill Book Company, Inc., New York, 1948.

Illustrative values of power factor of other materials are: quartz, mica, 0.0005; glass, steatite, 0.005; phenol products, 0.05.

5. Attenuation. A formula, good at high frequency, for attenuation owing to conductor resistance is given by Eq. 12-9. Equation 12-17 can now be substituted for r, and Eq. 12-4 for Z_0, to give a formula for *attenuation in a coaxial line owing to conductor resistance*:

$$\alpha = \frac{K}{276} \frac{\sqrt{fk}}{b} \frac{\left(1 + \frac{b}{a}\right)}{\log \frac{b}{a}} \tag{12-19}$$

where a and b are inner and outer radii, in the same unit of length
f is frequency (cycles per second)
k is relative dielectric constant of insulation
K is a factor from Eq. 12-17
α is per unit length, unit of length being same as for a and b

The corresponding equation for *attenuation in a two-wire line in air owing to conductor resistance* is

$$\alpha = \frac{K}{276} \frac{\sqrt{f}}{b} \frac{b/a}{\log b/a} \tag{12-20}$$

where a and b are wire radius and spacing, respectively, and other symbols have the same meaning as above. Proximity effect is not included.

The part of the attenuation that results from dielectric loss is given by Eq. 12-10 as $\frac{1}{2}gZ_0$. Introducing g from Eq. 12-18 and Z_0 from 12-4, *attenuation owing to dielectric loss is*

$$\begin{aligned} \alpha = \frac{gZ_0}{2} = \frac{\omega c}{2}\sqrt{\frac{l}{c}}\,(\text{PF}) &= \frac{1}{2}\omega\sqrt{lc}\,(\text{PF}) \\ &= \tfrac{1}{2}\beta\,(\text{PF}) \\ &= \frac{\pi}{\lambda}\,(\text{PF}) \\ &= \frac{\pi}{\lambda_0}\sqrt{k}\,(\text{PF}) \end{aligned} \tag{12-21}$$

where (PF) is power factor of the dielectric
λ is wavelength in the dielectric (meters)
λ_0 is wavelength of the same frequency in air (meters)
k is relative dielectric constant
α is attenuation per unit of length, the unit of length being the same as is used in l, c, β, λ, or λ_0

Several expressions for α are given in Eq. 12-21, and whichever is most convenient may be used. It is interesting that attenuation owing to dielectric hysteresis is quite independent of line dimensions, and depends only on the quality of the dielectric material.[3]

Total attenuation is the sum of that resulting from conductor resistance (Eq. 12-19 or 12-20) and that resulting from dielectric loss (Eq. 12-21).

Attenuation is given by these equations in *nepers*. This is readily changed to decibels (Eq. 9-10) to obtain a measure of power loss. Thus for a *coaxial line terminated in its characteristic impedance,*

$$\text{Power loss} = 8.686\alpha = \frac{K}{31.8}\,\frac{\sqrt{fk}}{b}\,\frac{1 + b/a}{\log_{10} b/a} + \frac{27.3\sqrt{k}}{\lambda_0}\,(\text{PF}) \quad \text{decibels per unit length} \qquad (12\text{-}22)$$

This equation will be found to be the most convenient for many practical computations of loss.

Optimum Dimensions. For a coaxial line of a given outside diameter, there is an optimum size of wire for which attenuation is least. That is, there is a certain value of b/a (b being fixed) that gives minimum α in Eq. 12-19. It is evident that this must be so, for if b/a approaches its minimum value of 1, the logarithm in the denominator of the expression for α approaches zero and α becomes large without limit; if, on the contrary, b/a becomes very large, the numerator containing b/a becomes large faster than the denominator which contains log b/a, and again α becomes large without limit. There must be a minimum value of α between these limits.

Physically, if the inner conductor is smaller than the optimum size, its resistance is higher and loss is increased. If the inner conductor is larger than optimum, the increased capacitance lowers the value of Z_0 and hence more current is required to transmit a certain amount of power, with the result that loss is again increased.

Attenuation owing to dielectric hysteresis need not be considered in this discussion, for such loss, as is shown by Eq. 12-21, is independent of line dimensions and hence does not affect the optimum ratio of b/a. (If the dielectric loss resulted from actual conductivity of the material, with g independent of frequency, instead of being due to dielectric hysteresis, there would be a slight but unimportant change of the optimum ratio.)

Although the optimum value of b/a is 3.6, this is not a critical value. Attenuation is less than 10 per cent above the minimum if the value of b/a is anything between 2.3 and 6.

---

[3] Some authors define the loss in a dielectric material by using the quality factor $Q = \omega\epsilon\rho$ as a characteristic of the material (ϵ is absolute dielectric constant, ρ is resistivity). For all practical purposes, $Q = 1/(\text{PF})$ (see footnote 9, Chap. 7), and Q may be used in Eq. 12-21 if desired.

Z_0 for Minimum Attenuation. Corresponding to the optimum ratio of b/a, there must be an optimum value of characteristic impedance. For air dielectric, Eq. 12-4 gives $Z_0 = 138 \log b/a$, and if $b/a = 3.6$, $Z_0 = 78$ ohms.

Hence, a coaxial line of given outside diameter, air-insulated, has minimum attenuation if the characteristic impedance is between 77 and 78 ohms. This is an important reason for the widespread practical use of lines with approximately this impedance.

However, a line designed for minimum attenuation is not best for all purposes. A line may be designed to transmit maximum power. The limiting factor is electric field strength at the surface of the inner conductor; if a critical value of field strength of about 30,000 volts per centimeter is exceeded, corona or sparking results. From Eqs. 6-70 and 6-71, voltage between the inner and outer conductors of a coaxial line is $V = E_s a \ln b/a$, E_s being the electric field at the surface of the inner conductor. Power transmitted is V^2/Z_0. Recognizing that E_s has a maximum value, there is an optimum value of b/a to give maximum power transmission. This is 1.65, and the corresponding characteristic impedance is 30 ohms.

A line may be designed to act as a resonant circuit. A short-circuited quarter-wavelength line has high input impedance, and an open-circuited quarter-wavelength line has low input impedance. These will be discussed in the next section. For a short-circuited resonant coaxial line to have maximum impedance, b/a must be 9.2, corresponding to $Z_0 = 133$ ohms for an air-insulated line. For an open-circuited resonant line to have minimum impedance, the inner conductor of the coaxial line should be as large as possible, which requires that Z_0 approach zero.

Lines with characteristic impedance of 50 to 80 ohms are common in practice.

For *two-wire lines*, there are similar optimum values of b/a if b, the spacing, is held constant. Working from Eq. 12-20, α is minimum if $b/a = 2.7$. This neglects proximity effect which is quite appreciable for such a small value of b/a. When proximity effect is included, the optimum b/a is about 4. Attenuation is less than 10 per cent above the minimum if b/a is between 3 and 6.5. The practical value of b/a for a two-wire line is usually above this optimum range. A ratio of 4 leaves a clear space between conductors that is only as great as the conductor diameter. It is more common to have b/a equal 10 or more.

6. Lines as Circuit Elements. Transmission lines find frequent use at radio frequency as circuit elements because of their desirable impedance properties. Lines that are used for such purposes are usually open-circuited or short-circuited at the receiving end, and do not serve to transmit energy—the term "transmission line" is preserved only by custom.

Because the particular interest is in impedance, Eq. 5-32 is helpful:

$$\frac{Z}{Z_0} = \frac{(Z_r/Z_0)\cos Bx + j\sin Bx}{\cos Bx + j(Z_r/Z_0)\sin Bx} \tag{12-23}$$

It will be recalled that $B = \beta - j\alpha$ and, if line losses can be neglected, Eq. 12-23 may be written in terms of β instead of B. The equation is readily expanded and simplified for various applications.

For instance, for a short-circuited line, $Z_r = 0$, and Eq. 12-23 becomes

$$\frac{Z}{Z_0} = j\tan Bx \tag{12-24}$$

For an open-circuited line, Z_r is infinite and

$$\frac{Z}{Z_0} = -j\cot Bx \tag{12-25}$$

If a line is just $\frac{1}{4}$ wavelength long, $\beta x = \pi/2$. If losses may be neglected, so that we may write $Bx = \pi/2$, Eq. 12-23 becomes

$$\frac{Z_s}{Z_0} = \frac{Z_0}{Z_r} \tag{12-26}$$

Z_s is the input or sending-end impedance of the line, and it is thus related to the load or receiving-end impedance by the simple and useful relation

$$Z_sZ_r = Z_0^2 \tag{12-27}$$

This equation is applicable to a *quarter-wavelength line* with any termination except open circuit or short circuit. An open or short-circuited line acts like a resonant circuit, and line loss is an essential factor that cannot be neglected.

7. Lines as Resonant Elements. If we consider only what appears at the input terminals, a *short-circuited quarter-wavelength line* and a parallel-resonant circuit of coil and condenser have these characteristics in common: both present extremely high impedance to one particular frequency; with both, the impedance at resonance is resistive; with both, the impedance drops rapidly if frequency varies slightly from resonance; both have inductive reactance at frequencies below resonance and capacitive reactance at frequencies above resonance; both will carry direct current freely while effectively blocking the frequency to which they are resonant. An inherent difference is that the transmission line displays similar resonance at all odd multiples of its lowest resonant frequency, and has inverse resonance characteristics (as of a short-circuited half-wavelength line) at the even multiples.

An *open-circuited quarter-wavelength line* is similar to a series-resonant circuit of coil and condenser. It has extremely low impedance at the resonant frequency; it is resistive at resonance, inductive above, and capacitive below. It blocks direct current, while passing freely the resonant frequency. Like a short-circuited line, but unlike a circuit of lumped constants, its charactersitics tend to repeat at odd multiples of the lowest resonant frequency, whereas at even multiples the inverse characteristics appear.

The selectivity, or frequency sensitivity, of a resonant circuit is usually measured by the quality factor Q. There may be various definitions[4] for Q, and for lumped circuits it is immaterial which is adopted. With lines, however, the possible definitions are not always coincidental, and it will be best to base the following discussion on the one that is most frequently a matter of practical interest: the frequency sensitivity near resonance.

Frequency sensitivity of a lumped parallel-resonant circuit is as follows. If Z is the input impedance at any operating frequency (limited, however, to a frequency near resonance, and to a circuit with fairly low losses and consequent high Q) and Z_{res} is the input impedance at resonance, it is shown in books on circuit theory that

$$\frac{Z}{Z_{\text{res}}} = \frac{R}{R + j(\omega L - 1/\omega C)} \tag{12-28}$$

where R is the resistance of the circuit of coil and condenser (measured around the loop)
L is the inductance of the coil
C is the capacitance of the condenser
ω is 2π times the operating frequency

Let us specify further that
ω_0 is the value of ω at resonance
$\delta\omega$ is the tuning off resonance, $\delta\omega = \omega - \omega_0$

Then, noting that $\omega_0 L = 1/\omega_0 C$, by definition of resonance, it is a good approximation that

$$\frac{Z}{Z_{\text{res}}} = \frac{R}{R + j2L\,\delta\omega} = \frac{R}{R + j2\dfrac{\delta\omega}{\omega_0}\dfrac{\omega_0 L}{R}R} \tag{12-29}$$

Using Q to represent $\omega_0 L/R$, this becomes

$$\frac{Z}{Z_{\text{res}}} = \frac{1}{1 + j2Q\dfrac{\delta\omega}{\omega_0}} \tag{12-30}$$

---

[4] Based on ratio of reactance to resistance; total current (or voltage) to current (or voltage) of one element; ratio of stored energy to energy consumed per cycle; or frequency selectivity as discussed in the text.

(This is approximately the formula for the *universal resonance curve.*) Among the many useful deductions from this equation are the following:

1. When the detuning ratio, $\delta\omega/\omega_0$, is $1/2Q$, the impedance is less than the impedance at resonance by the factor $1/\sqrt{2}$, or the impedance is approximately 70 per cent of impedance at resonance.

2. With this same amount of detuning, the angle of the impedance is 45 degrees; thus its components of resistance and reactance are equal.

For comparison, the frequency sensitivity of a short-circuited quarter-wavelength line may be analyzed. Referring to Eq. 12-24,

$$\frac{Z}{Z_0} = \frac{j \sin Bx}{\cos Bx} = j\,\frac{\sin \beta x \cosh \alpha x - j \cos \beta x \sinh \alpha x}{\cos \beta x \cosh \alpha x + j \sin \beta x \sinh \alpha x} \tag{12-31}$$

For quarter-wave resonance, $\beta x = \pi/2$. For frequencies near resonance, $\sin \beta x$ is approximately 1 and $\cos \beta x$ is much smaller than 1. For a line with low losses, α is much smaller than β; $\sinh \alpha x$ is very small and is approximately αx, whereas $\cosh \alpha x$ is approximately 1. Using these approximations in Eq. 12-31,

$$\frac{Z}{Z_0} \approx \frac{1}{\alpha x - j \cos \beta x} \tag{12-32}$$

Exactly at resonance, $\cos \beta x = 0$ and hence $Z_{\text{res}}/Z_0 = 1/\alpha x$. Slightly off resonance, $\cos \beta x$ is small and is approximately proportional to the amount of detuning from resonance; to show this, we write

$$\begin{aligned} \cos \beta x &= \cos \beta_0 \frac{\omega}{\omega_0} x = \cos \beta_0 \frac{\omega_0 + \delta\omega}{\omega_0} x \\ &= \cos\left(\frac{\pi}{2} + \beta_0 \frac{\delta\omega}{\omega_0} x\right) \\ &= -\sin \beta_0 \frac{\delta\omega}{\omega_0} x \approx -\beta \frac{\delta\omega}{\omega_0} x \end{aligned} \tag{12-33}$$

The above equation proceeds in five steps. In the first, using subscript 0 to indicate conditions at resonance, it is recognized that β is proportional to ω. In the second, the tuning off resonance $\delta\omega = \omega - \omega_0$ is introduced. In the third, it is noted that $\beta_0 x = \pi/2$, this being the value of βx at resonance. The cosine is then changed to the sine, and since $\delta\omega/\omega_0$ is small it is a good approximation to let the sine equal its argument.

We have now evaluated Eq. 12-32 at resonance, and also slightly off resonance. The ratio is

$$\frac{Z}{Z_{\text{res}}} = \frac{\alpha x}{\alpha x + j\beta \dfrac{\delta\omega}{\omega_0} x} = \frac{1}{1 + j \dfrac{\beta}{\alpha} \dfrac{\delta\omega}{\omega_0}} \tag{12-34}$$

If we now write

$$Q = \frac{\beta}{2\alpha} \tag{12-35}$$

the equation becomes

$$\frac{Z}{Z_{res}} = \frac{1}{1 + j2Q\dfrac{\delta\omega}{\omega_0}} \tag{12-36}$$

The similarity of Eq. 12-36 to 12-30 shows two things. First, the *universal resonance curve* applies as a good approximation to the variation of impedance of a quarter-wavelength transmission line near resonance (the fundamental reason being that the curve is a good approximation of the complex tangent function in that restricted region). Second, the frequency sensitivity and selectivity of a quarter-wavelength line is characterized by the quantity $\beta/2\alpha$ in the same way that the response of a lumped resonant circuit is characterized by Q.

Hence, as in Eq. 12-35, $\beta/2\alpha$ is commonly spoken of as the Q of a transmission line. The two statements regarding Q that follow Eq. 12-30 can also be applied to a resonant line, based on Eq. 12-36. Attention may also be called to the fact that $Q = \omega l/r$ for a line with no dielectric loss, this being analogous to the Q of a coil.

As has been mentioned, a short-circuited line is resonant at all odd multiples of the frequency for which it is $\frac{1}{4}$ wavelength long, and at these higher frequencies it displays similar properties. The Q at each resonant frequency is determined by the values of β and α that apply at that frequency. The value of β is proportional to frequency. If line loss is predominantly in the resistivity of the metal conductors, α is proportional to the square root of frequency (Eq. 12-19); if loss is primarily in the dielectric material, α is proportional to frequency (Eq. 12-21). Hence, for a line of given length, if the line is in air, Q at the higher resonant frequencies will tend to increase as the square root of the frequency; if the line has solid dielectric material, Q will be approximately constant.[5]

8. Impedance at Resonance. Q indicates the sharpness of resonance; it also indicates the maximum impedance available from a *short-circuited line at resonance.* It was deduced from Eq. 12-32 that the impedance at resonance is

$$Z_{res} = \frac{Z_0}{\alpha x} = \frac{2Q}{\beta x} Z_0 \tag{12-37}$$

---

[5] At a wavelength in the neighborhood of 10 centimeters, 1,000 is not an unreasonable value for the Q of a solid-dielectric line, and 10,000 for the Q of a line in air.

Since resonances occur when $\beta x = \pi/2$, $3\pi/2$, $5\pi/2$, and so on, this may be written

$$Z_{\text{res}} = \frac{4Q}{n\pi} Z_0 \tag{12-38}$$

where n is the number of quarter-wavelengths, an odd integer.

It is evident that, on a line of given length, Z_{res} is less high at the higher resonant frequencies, corresponding to larger n's. Even if Q increases as the square root of frequency, as on a line with no dielectric losses, the n in the denominator increases more rapidly. If Q is substantially independent of frequency, as on a line with high dielectric loss, Z_{res} at the higher resonant frequencies has only one-third, one-fifth, one-seventh, and so on, of its value at the lowest resonant frequency.

If the length of a line is to be determined to give resonance at a given frequency, it is always advantageous to use a quarter-wavelength line rather than a three-quarter-wavelength or longer line. Q is the same, at the same frequency, for various lengths of line (see Eq. 12-35), but the maximum obtainable resistance, as in Eq. 12-38, is inversely proportional to the number of quarter-wavelengths.

The characteristics of an *open-circuited quarter-wavelength line* are comparable with those of the short-circuited quarter-wavelength line just discussed. The selectivity of the circuit is the same; the *universal resonance curve* is applicable to the *admittance* of the open-circuited quarter-wavelength line near resonance, and Eq. 12-36 applies to the ratio of admittances Y/Y_{res} (instead of Z/Z_{res}). The same statements can be made about the selectivity and frequency sensitivity, but applying to admittance instead of impedance. Working from Eq. 12-25, for a low-loss open-circuited line near resonance,

$$\frac{Z}{Z_0} = \frac{\cos\beta x \cosh\alpha x + j\sin\beta x \sinh\alpha x}{j\sin\beta x \cosh\alpha x + \cos\beta x \sinh\alpha x} \approx \alpha x - j\cos\beta x \tag{12-39}$$

For comparison with Eq. 12-32:

$$\frac{Y}{Y_0} \approx \frac{1}{\alpha x - j\cos\beta x} \tag{12-40}$$

Y_0, the characteristic admittance of the line is $1/Z_0$, or approximately $\sqrt{c/l}$.

Since the admittance of an open-circuited line for which $\cos\beta x = 0$ is seen from Eq. 12-40 to be

$$Y_{\text{res}} = \frac{Y_0}{\alpha x} = \frac{2Q}{\beta x} Y_0 \tag{12-41}$$

it follows that admittance at the various resonant frequencies corresponding to odd multiples of quarter-wavelength operation can be expressed as

$$Y_{\text{res}} = \frac{4Q}{n\pi} Y_0 \tag{12-42}$$

where n is an odd integer.

A high-impedance selective resonant circuit is frequently required, and a line may be used for this purpose in either of two ways: an open-circuited half-wavelength line has the desired characteristics, and so has a short-circuited quarter-wavelength line. Let us compare the two possibilities.

They are equally selective, for both have the same Q.

The maximum impedance of the short-circuited quarter-wavelength line is, by Eq. 12-38, $4QZ_0/\pi$. The maximum impedance of the open-circuited line is found from the second member of Eq. 12-39 by introducing the condition that the line is $\frac{1}{2}$ wavelength long; that is, $\beta x = \pi$ or $x = \pi/\beta$. This gives $Z = Z_0/\alpha x = \beta Z_0/\alpha\pi = 2QZ_0/\pi$, which is just half the impedance of the short-circuited quarter-wavelength line.

Various other factors may be of practical importance. The short-circuited quarter-wavelength line will carry direct and low-frequency currents; the open-circuited half-wavelength line will block them. At radio frequencies other than the desired resonant frequency, such as half and double the fundamental resonant frequency, the open and short-circuited lines have quite different characteristics; this may be important in connection with harmonics.

The length of a short-circuited line is more readily adjustable than is the length of an open-circuited line, and also, if coaxial, the short-circuited line is more completely shielded and self-contained.

It is such factors as these that determine whether a short-circuited or open-circuited line should be used in practical application. We note again an illustration of the general principle that the shorter line always has better impedance characteristics because of having lower total loss.

Summary. From the above derivations, the impedance of a resonant line for which the length is n quarter-wavelengths is given by

$$\frac{Z}{Z_0} = \frac{4Q}{n\pi} \begin{cases} \text{for short-circuited line with } n = 1, 3, 5, \ldots \\ \text{for open-circuited line with } n = 2, 4, 6, \ldots \end{cases}$$

$$\frac{Y}{Y_0} = \frac{4Q}{n\pi} \begin{cases} \text{for short-circuited line with } n = 2, 4, 6, \ldots \\ \text{for open-circuited line with } n = 1, 3, 5, \ldots \end{cases}$$

where $Q = \dfrac{\beta}{2\alpha} = \dfrac{\pi\sqrt{k}}{\alpha\lambda_0}$

λ_0 is wavelength of the same frequency in air (meters)
k is relative dielectric constant of the line
α is nepers per meter, from Sec. 5

Optimum Design. In Sec. 5 it was seen that there is an optimum design of line to give minimum attenuation. There is also an optimum ratio of b/a to give maximum input impedance at resonance. Impedance is $Z_{res} = Z_0/\alpha x$, and since Z_0 and α are both functions of b/a, the maximum resonant impedance does not correspond to the minimum attenuation.

Let us find the optimum value for a line without dielectric loss. (If dielectric loss were considerable, the optimum ratio would be affected, but it is not probable that a line with dielectric loss would be used as a high-impedance resonant circuit.) Substituting Eqs. 12-35, 12-4, 12-7, and 12-19 into Eq. 12-38, the impedance at resonance of a short-circuited coaxial line is

$$Z_{res} = \frac{5{,}080}{Kn} 10^{-9} \sqrt{f}\, b \frac{(\log b/a)^2}{1 + b/a} \tag{12-43}$$

where a and b are inner and outer radii of the line (centimeters)
f is frequency (cycles per second)
n is the number of quarter-wavelengths, an odd integer
K is given with Eq. 12-17

If the line conductors are copper, $K = 41.6 \times 10^{-9}$ and

$$Z_{res} = \frac{122\sqrt{f}\, b}{n} \frac{(\log b/a)^2}{1 + b/a} \tag{12-44}$$

The optimum b/a to give maximum impedance in Eq. 12-43 or 12-44 is about 9.2. Thus maximum impedance of a coaxial line is obtained with a much smaller inner conductor than is maximum Q. However, the impedance maximum is quite broad, and the resonant impedance is within 10 per cent of the maximum value if b/a is between about 5 and 20.

The optimum b/a to give maximum impedance to a short-circuited quarter-wavelength line of two parallel wires is about 8.0.

Equations 12-43 and 12-44 apply also to an open-circuited line that is $\frac{1}{2}$ wavelength long, or some multiple of this length. The integer n is then even. The optimum b/a is of course the same.

9. Resonant Voltage Rise. A series-resonant circuit of coil and condenser can be used to provide an increased voltage; the voltage across either the coil or the condenser is greater in magnitude than the applied voltage by a factor of approximately Q. An open-circuited quarter-wavelength line can also be used to raise voltage; the voltage at the open end of the line is greater than the applied voltage. From Eq. 4-8, letting Z_r be infinite,

$$\frac{V_s}{V_r} = \cos Bx = \cos (\beta - j\alpha)x \tag{12-45}$$

For resonance, βx is to be an odd multiple of $\pi/2$; hence the length x is $n\pi/2\beta$ with n an odd integer, and

$$\frac{V_s}{V_r} = \cos\left(\frac{n\pi}{2} - j\frac{n\pi\alpha}{2\beta}\right) = \pm\sin j\frac{n\pi}{4Q} \tag{12-46}$$

Since Q is large, the argument is small and is approximately equal to the sine. The amplitude of voltage increase at the open receiving end of the line that is n quarter-wavelengths long is hence

$$\left|\frac{V_r}{V_s}\right| = \frac{4}{n\pi}Q \tag{12-47}$$

This compares with the voltage increase of Q obtained from a resonant circuit with lumped parameters; the voltage increase is slightly higher than Q for a $\frac{1}{4}$-wavelength line, but if the line is 3 or 5 quarter-wavelengths the resonant rise of voltage at the end is greatly reduced. In comparing with the voltage to be obtained from a circuit of coil and condenser, it is to be remembered that the Q of a line is greatly higher than the Q of such a circuit at a high radio frequency, and the Q of a line improves with increasing frequency.

Practical Considerations. In concluding the discussion of resonant lines it should be mentioned that the practically obtainable Q is never as high as the Q predicted from the equations of this section, particularly for frequencies of centimeter waves. The losses are always greater than the formulas give. There are several contributing causes. Any imperfection of the line and associated apparatus increases loss. Because of extreme skin effect, roughness of the surface of the conductors is important; so, also, is the porosity that may result from silver plating. A thin layer of metallic oxide on the conductor surfaces will increase the loss, for current is driven into such a surface layer by skin effect. The use of silver plating is advantageous in avoiding semiconducting surface oxides.

Any small loss of energy by radiation will lower the Q, and radiation is not taken into account in the above equations. A possible example is radiation from the open end of an open-circuited coaxial line. The resistance of the short-circuiting disc or plunger in a short-circuited line has also been omitted.

In a system with moderate Q, the actual value of Q may come close to the predicted value. In a system specially designed for high Q, however, a very small amount of loss will make an appreciable difference, and the actual Q that can be obtained may turn out to be somewhere between the predicted Q and half the predicted Q.

A matter of importance in the consideration of all short-circuited lines is the length of the short-circuiting member itself. It is evident that the short-circuiting bridge on a two-wire line is similar to an additional length of line. If the bridge is a wire or rod, connecting the ends of the wires as in Fig. 12-3*a*, it will have magnetic and electric fields about it somewhat similar to those about the main conductors. To speak a little more precisely, the electromagnetic field of the line will be distorted near the short-circuiting wire. As a fair approximation, the effective length of line is the actual length of the line increased by approximately half the wire-to-wire spacing.

If, however, the line is short-circuited by a large disc of copper or other good conducting material, as in Fig. 12-3*b*, there is relatively little distortion of the electromagnetic field at the end of the line. It may be considered that the incident wave is reflected by the large disc, as by a mirror, with hardly any necessity for readjustment of the geometry of the fields. There is little electric or magnetic field beyond the disc. Hence the actual length of the line is a very fair measure of its equivalent electrical length.

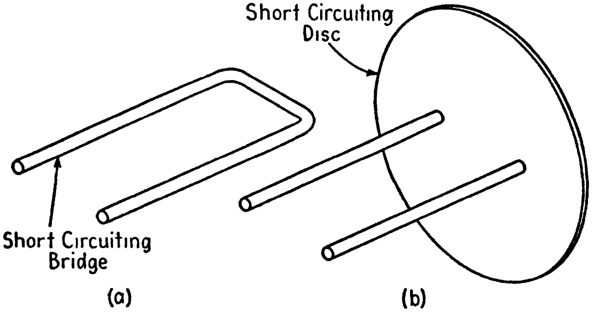

FIG. 12-3. Types of short circuit.

A coaxial line may be short-circuited by a disc that closes all the space between inner and outer conductors. It may also be short-circuited by a single wire from the inner to the outer conductor. The latter is much less satisfactory; it destroys the symmetry of the coaxial line, it leaves the end open to stray fields and permits the line fields to escape from the end of the tube to some extent, and it makes the effective length of the line somewhat indefinite (see also Sec. 13, Chap. 13).

If a coaxial line is closed with a short-circuiting disc, there should be good electrical connection between the disc and the line conductors around the entire circumference. Bad contact, permitting current to flow but resulting in loss, is harmful to the resonant properties of the line. Contact resistance occurring at the point of short circuit, where current is maximum, is more disturbing than would be an equal amount of resistance at any other point in the line (see Sec. 10).

10. Examples of Quarter-wavelength Resonance. Three facts relating to quarter-wavelength lines are used so often that they should be thoroughly familiar. They have previously been derived and will merely be summarized at this point:

1. The input impedance to a *short-circuited* quarter-wavelength line is an exceedingly high resistance, approaching infinity if loss is negligible.

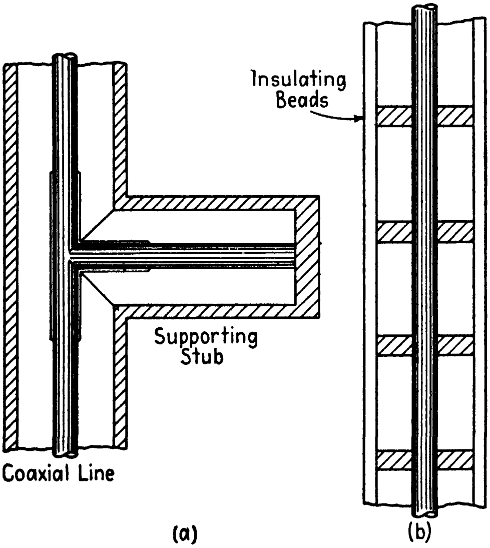

FIG. 12-4. Coaxial line, showing means of supporting inner conductor.

2. The input impedance to an *open-circuited* quarter-wavelength line is a very low resistance, approaching zero if loss is negligible.

3. More generally, for a quarter-wavelength line, neglecting loss,

$$Z_s Z_r = Z_0^2 \tag{12-48}$$

where Z_s is sending-end or input impedance
Z_r is receiving-end or load impedance
Z_0 is characteristic impedance of the line

Supporting Stub. An example of the use of a quarter-wavelength line as a high impedance is the supporting stub shown in Fig. 12-4*a*. The support of the inner conductor of a coaxial line at centimeter-wave frequencies is a serious design problem. The use of solid dielectric material as a support or spacer, even in the form of beads at intervals along the line as in Fig. 12-1, is undesirable because of dielectric loss in the electric field. A practical solution in some cases is to use a metal rod to support the line, as shown; the metal rod is itself the inner conductor of a short coaxial line, extending at right angles to the main line, and short-circuited at its outer end. By adjusting the length of this supporting stub line so that it is electrically equivalent to a quarter-wavelength long (the geometry of the supporting tee being taken into account) it appears as an almost infinite impedance to a wave on the main line. Actually this all-metal support can be constructed to divert less energy from the main line than would be lost in an adequate support of dielectric material.

A quarter-wavelength supporting stub in its simplest form is highly frequency-sensitive, for the stub line does not have high impedance except at one particular frequency. Indeed, the better it is, and the higher its impedance, the more sensitive it is to frequency change. By proper design of the sleeve that holds the main line, the support can be made much less frequency-sensitive, however.

Supporting Beads. The simplest means of supporting the inner conductor of a coaxial line is to hold it in place with solid dielectric material. To avoid unnecessary dielectric loss, the solid dielectric may take the form of beads or discs spaced along the line, as in Fig. 12-4*b*. (Flexible coaxial line is completely filled with solid dielectric material, but rigid line need not be.) The effect of such beads depends on whether their spacing bears any simple relation to the wavelength transmitted.

If the wavelength is much greater than the bead spacing, or if the spacing is random, or unrelated to the wavelength, the beads merely increase the effective dielectric constant. If the thickness of the bead is w, the relative dielectric constant k, and the spacing s, the effective dielectric constant of the line is the average value

$$1 + (k - 1)\frac{w}{s}$$

See the coaxial line and shielded pair in Fig. 12-1.

If the beads are spaced $\frac{1}{4}$ wavelength apart (including the fact that the wavelength is shortened by the dielectric constant of the bead), it is best to consider the repeated small reflections that take place when a wave passes a bead. There will be some reflection from each surface of each bead, owing to the abrupt change of Z_0 as the dielectric constant changes. Reflection as the wave enters the bead will be opposite in sign to reflection as the wave leaves the same bead, and the thinner the bead the more nearly these two reflections from the same bead will add to zero. However, there will be some residual reflection from a bead. There will also also be reflection from the next bead.

If the beads are $\frac{1}{4}$ wavelength apart, the reflection from the second bead will be just out of phase with the reflection from the first bead, as $\frac{1}{2}$ cycle will elapse while the incident wave travels the distance between beads and the reflected wave returns. Since the reflection from the first bead and the reflection from the second bead are practically equal in magnitude and opposite in phase, they will cancel and there will be no net reflection. Quarter-wavelength spacing is therefore desirable, but it is somewhat sensitive to frequency and is not always satisfactory for *broad-band* operation.

On the other hand, if the beads are $\frac{1}{2}$ wavelength apart, the reflection from the second bead will add in phase with the reflection from the first bead. Reflections from all other beads will add in a cumulative manner, producing very serious reflection in a long line. Thus a half-wavelength spacing (or spacing of approximately $\frac{1}{2}$ wavelength, if losses are taken into account) is extremely undesirable.

In general, repeated discontinuities of like kind should be spaced at odd multiples of $\frac{1}{4}$ wavelength. Discontinuities of unlike kind (such as change from air to solid dielectric, followed by change from solid dielectric back to air) should be multiples of $\frac{1}{2}$ wavelength apart. For instance, a dielectric bead $\frac{1}{2}$ wavelength thick produces little net reflection.

Resonant Short-circuiting Devices. Good contact is important in the design of sliding plungers that are intended to adjust the position of the short circuit in a coaxial line. A plunger sliding along a line, making perfect contact with both inner and outer conductors at all points, could be used to adjust the length of the short-circuited line. Despite the most careful design, however, the losses resulting from bad contact in sliding devices are serious.

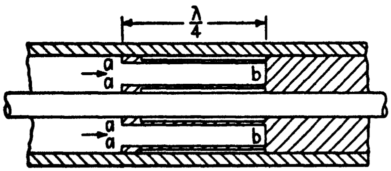

FIG. 12-5. Spring-finger type of sliding contact.

An improvement of the shorting plug results from attaching to the plug, as at *b* in Fig. 12-5, sets of spring fingers that effectively form a pair of inner and outer tubes extending from *b* to *a*. The distance from *b* to *a* is made exactly $\frac{1}{4}$ wavelength. This device gains some advantage from having springy fingers that make good contact at *a*. A greater advantage is in having the sliding contact at a point, $\frac{1}{4}$ wavelength from the short circuit, where the current is *minimum* and resistance causes the least loss of power. A major disadvantage of the device is that it uses a line that can never be less than

$\frac{1}{4}$ wavelength long; this results in unnecessarily large line loss. Also, the spring fingers are not easily kept in adjustment and contact resistance is rather variable, depending upon alignment and surface finish.

A more effective device[6] is shown in Fig. 12-6. The general construction of this "half-wave trap" is shown in Fig. 12-6*a*, and a detail in Fig. 12-6*b*. Referring to the detail, a wave coming from the left is effectively short-circuited and reflected at the surface *a*. The plunger does not quite make contact with the line conductor at the short-circuiting face *a*, for a small but definite clearance is left at *b*. There is a narrow passage from *b* to *c*. This passage is actually a coaxial transmission line of very close spacing.

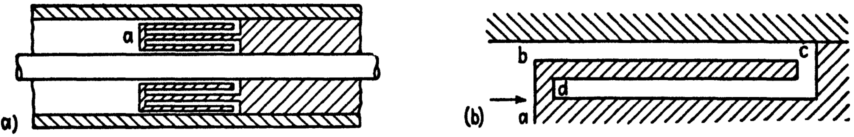

FIG. 12-6. A sliding plunger with "half-wave trap."

By proper design, input impedance to this *trap* line at *b* can be made practically zero, thereby completing the short circuit of the main line at the face *ab*.

The distance from *b* to *c* is $\frac{1}{4}$ wavelength. The distance from *c* to *d* is also $\frac{1}{4}$ wavelength. Some small part of the wave impinging on the surface *ab* will enter the trap at *b*; it will pass on to *c* and to *d*, where it will be reflected back to *c* and to *b* again. The passage *bcd* is a half-wavelength short-circuited line, and its input impedance (at *b*) is practically zero, as desired.

There should be good contact between plunger and tube at the point *c*. However, this is a point of minimum current, for *c* is $\frac{1}{4}$ wavelength from the short circuit at *d*, and contact resistance is less serious here than elsewhere.

The quality of a resonant line is inversely proportional to power loss. A *half-wave trap* is effective because it introduces a minimum of power loss in the system. Only a small fraction of the total power in the main line enters the *trap* line, and only a small fraction of this small fraction is lost (in resistivity of the metal walls or in contact resistance at *c*), the rest being returned to the main line after reflection at *d*. Actual performance of the half-wave trap is quite satisfactory.

This device is discussed in some detail, not only because of its practical importance, but even more because it illustrates the general principles mentioned on page 322.

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[6] See Moreno (Ref. 20).

Quarter-wavelength Line for Impedance Matching. Another example of interest, an illustration of the use of Eq. 12-48, is the impedance transformation produced by a quarter-wavelength line. A load, as in Fig. 12-7, receives energy from a line to which its impedance is not matched. Let us assume that the load is pure resistance (or, what is the same thing, that enough transmission line is included *within* the load so that a pure resistance appears when looking in at the terminals marked b). By properly designing a quarter-wavelength section of line, from a to b, the load can be matched to the main line.

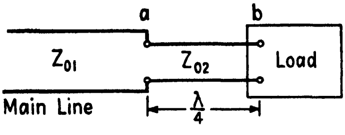

FIG. 12-7. A quarter-wavelength section for impedance matching.

The characteristic impedance of the section of line from a to b may be called Z_{02}, and Z_b is its load impedance. The impedance looking toward the load at a is required to be equal to the characteristic impedance of the main line (to provide a match) and may be called Z_{01}. By Eq. 12-48,

$$Z_{01}Z_b = Z_{02}^2 \tag{12-49}$$

Z_{01} and Z_b are given; Z_{02}, the characteristic impedance of the quarter-wavelength section, is designed to satisfy this relation, and thus to prevent resonance or standing waves on the main line. There will, of course, be resonant effects on the quarter-wavelength section; indeed, it is resonance along this section that provides the desired impedance transformation.

This is not a particularly advantageous way to match line and load, for the quarter-wavelength section is not easily adjustable in either Z_0 or length. With a two-wire line, Z_0 could be adjusted by changing the spacing between wires, but (despite the diagrammatic representation of the line in the figure) it is not likely that a two-wire line would be used for work at frequencies that make a quarter-wavelength section a practicable device. The characteristic impedance of a coaxial line is determined by the dimensions of the two conductors, and it cannot be changed after construction.

However, a fairly practical compromise is to have a line that would be a coaxial line except that the inner conductor is adjustable from its central position. By means of an eccentric mounting, it can be moved from the axis toward the wall of the tube. As the inner conductor moves toward the wall of the tube, the characteristic impedance of the line is decreased.

There are several practical methods of matching lines to loads at centimeter-wave frequencies. These use lines as impedance transformers and as reactive elements, and will be considered in Secs. 18 to 21.

11. Lines as Reactive Elements. Since inductance coils are unsatisfactory at centimeter-wave frequencies, it is often convenient to use a line that is terminated in either a short circuit or an open circuit as a reactive element. For example, a line may be used as inductance or capacitance in a filter network or in an impedance-matching network.

Although at exact resonance the input impedance to either an open- or short-circuited line is resistive, the impedance is almost purely reactive at intermediate frequencies. (From Eq. 12-36, impedance is largely reactive if the detuning, $\delta\omega/\omega_0$, is as much as five or ten times $1/Q$; and Q of a transmission line is so high that this is a relatively small deviation from resonance.)

The input reactance to a line may vary from zero to 70 per cent of the resonant maximum impedance, and may be either reactive or capacitive. If the length of the line is less than $\frac{1}{4}$ wavelength, the reactance of an open-circuited line is capacitive (negative); reactance of a short-circuited line is inductive (positive). If the line is between $\frac{1}{4}$ and $\frac{1}{2}$ wavelength long, an open-circuited line is inductive; a short-circuited line is capacitive.

To determine the line input reactance, the following are good approximations *unless* the length is near a multiple of $\frac{1}{4}$ wavelength. These are from Eqs. 12-25 and 12-24.

For an open-circuited line:

$$X = -Z_0 \cot \beta x = -Z_0 \cot \left(2\pi \frac{x}{\lambda}\right) \tag{12-50}$$

For a short-circuited line:

$$X = Z_0 \tan \beta x = Z_0 \tan \left(2\pi \frac{x}{\lambda}\right) \tag{12-51}$$

In these expressions x is length of line, and λ is wavelength. Near resonance the appropriate equations of Secs. 7 and 8, Eqs. 12-36 and 12-40, may be used to find reactance.

For some purposes it is necessary to know the resistance component of the input impedance to a line, even though the impedance is predominantly reactive. For a high-frequency line with small or moderate loss, *except* near frequencies that give resonant peaks of impedance, the resistance component is as follows:[7]

For an open-circuited line:

$$R = Z_0 \cot \beta x \left[\frac{2\alpha x}{\sin 2\beta x} - \frac{1}{2}\left(\frac{r}{\omega l} - \frac{g}{\omega c}\right)\right] \tag{12-52}$$

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[7] The derivation for the short-circuited line is as follows. It is assumed that $r \ll \omega l$, $g \ll \omega c$, $(\alpha x)^2 \ll 1$, $(\alpha x \tan \beta x)^2 \ll 1$. By Eq. 5-36,

$$Z = jZ_0 \tan Bx \tag{12-56}$$

For an open-circuited line with no dielectric loss ($g = 0$):

$$R = Z_0 \cot \beta x \cdot \frac{r}{2\omega l}\left(\frac{2\beta x}{\sin 2\beta x} - 1\right) \tag{12-53}$$

For a short-circuited line:

$$R = Z_0 \tan \beta x \left[\frac{2\alpha x}{\sin 2\beta x} + \frac{1}{2}\left(\frac{r}{\omega l} - \frac{g}{\omega c}\right)\right] \tag{12-54}$$

For a short-circuited line with no dielectric loss ($g = 0$):

$$R = Z_0 \tan \beta x \cdot \frac{r}{2\omega l}\left(\frac{2\beta x}{\sin 2\beta x} + 1\right) \tag{12-55}$$

Expand Z_0, retaining only first-order real and imaginary terms:

$$\begin{aligned} Z_0 &= \sqrt{\frac{r + j\omega l}{g + j\omega c}} = \frac{[\omega^2 lc + j\omega(lg - rc)]^{1/2}}{\sqrt{g^2 + \omega^2 c^2}} \\ &= \frac{1}{\sqrt{g^2 + \omega^2 c^2}}\left[(\omega^2 lc)^{1/2} + \frac{1}{2}(\omega^2 lc)^{-1/2} j\omega(lg - rc) + \cdots\right] \\ &\approx \sqrt{\frac{l}{c}} + \frac{j}{2\omega}\sqrt{\frac{l}{c}}\left(\frac{g}{c} - \frac{r}{l}\right) \end{aligned} \tag{12-57}$$

Expand $j \tan Bx$, noting that approximately $\tan \alpha x = \alpha x$,

$$\begin{aligned} j \tan Bx &= j \tan(\beta x - j\alpha x) \approx \frac{\alpha x + j \tan \beta x}{1 + j\alpha x \tan \beta x} \\ &\approx \alpha x(1 + \tan^2 \beta x) + j \tan \beta x \\ &= \frac{\alpha x}{\cos^2 \beta x} + j \tan \beta x \end{aligned} \tag{12-58}$$

Multiplying, by Eq. 12-56,

$$\begin{aligned} Z = R + jX &= \sqrt{\frac{l}{c}}\left[1 + \frac{j}{2\omega}\left(\frac{g}{c} - \frac{r}{l}\right)\right]\left[\frac{\alpha x}{\cos^2 \beta x} + j \tan \beta x\right] \\ &\approx \sqrt{\frac{l}{c}}\left[\frac{\alpha x}{\cos^2 \beta x} + \frac{\tan \beta x}{2\omega}\left(\frac{r}{l} - \frac{g}{c}\right) + j \tan \beta x\right] \end{aligned} \tag{12-59}$$

From the imaginary parts of Eq. 12-59, Eq. 12-51 results. Equating the real parts, and making a trigonometric substitution for $\cos^2 \beta x$, Eq. 12-54 is obtained. Equation 12-55 then results from letting $g = 0$ and $\alpha = r/2Z_0$. In Eqs. 12-50 to 12-55, the small angle of Z_0 is to be neglected, although this cannot be done in Eqs. 12-56 and 12-57. (Note, however, that for the distortionless line the imaginary component of Eq. 12-57 vanishes, Z_0 being purely real, and the parenthesis in Eqs. 12-52 and 12-54 then vanishes also.)

Derivation of the equations for the open-circuited line is parallel to the above, starting from Eq. 5-34.

12. Energy Transmission. When a line is operated at radio frequency for the purpose of transmitting energy, its principles of operation are much the same as for a telephone line. The object is to supply as much energy as possible to the load. Resonance is avoided by terminating the line in its characteristic impedance. The line is sometimes loaded to reduce attenuation.

There are two or three significant differences in radio-frequency operation, however. One is that the width of the band of frequencies to be transmitted is usually a much smaller fraction of the average frequency than is the case of an audio-frequency line.[8] Since a typical radio-frequency line may carry a frequency of several million cycles plus or minus only a few thousand, with a total band width of less than 1 per cent of the average or carrier frequency, it is unnecessary to give great attention to the variation of line characteristics with frequency. It is often sufficient, for instance, for the line and load to be *matched* at one frequency. Distortion, which is the result of variable response at different frequencies, is usually not much of a problem.

At radio frequency it is necessary to consider the radiation of energy from a line into space. An open-wire line, acting like an antenna, loses a fraction of the energy that it receives, and to avoid radiation a coaxial line is often used. The open-wire line may be said to have a certain *radiation resistance*; this is a hypothetical resistance which, multiplied by the square of the rms line current, I^2, gives radiated power. Careful consideration of radiation from a transmission line is a moderately involved problem in field theory and it must suffice here to say that the radiation resistance of a nonresonant two-wire transmission line is of the order of magnitude of $1{,}580\ (b/\lambda)^2$, where b is spacing and λ is wavelength (in the same units of length). This value may be doubled when radiation from connections at the ends of the line is considered. It applies fairly well to an open-circuited line even though resonant, but the radiation resistance of a short-circuited resonant line may be greatly lower. (For a resonant line, radiation resistance is multiplied by the square of the rms current at a point on the line where the current is maximum to obtain radiated power.) It is assumed that b is much smaller than λ and that the length of the line is many times b. It is interesting that the length of the line does not appear in this formula for radiation resistance, indicating that energy radiated is not primarily a function of length of line.

About the only practical conclusion that can be drawn from this approximate expression is that the value of radiation resistance which it gives should be low compared to Z_0 of the line. This will be true if the spacing is less than $\frac{1}{100}$ wavelength, a proportion that is easily attained in the broadcast frequency range but that is difficult with wavelengths of a few centimeters. Hence coaxial lines are almost always used for centimeter waves in preference to two-wire lines.

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[8] *Video-frequency operation*, although employing higher frequencies, is obviously comparable to *audio-frequency operation* for purposes of this discussion. It has the same problems, many being accentuated.

If coaxial lines are open at the ends, or are closed by discs, or in some other way the axial symmetry of the system is reasonably well preserved, radiation is negligibly small. However, if the terminating equipment is connected without regard for axial symmetry, radiation may be comparable to that from a two-wire line.[9]

The major problem at radio frequency, however, is the same as at audio frequency: if a line is intended to transmit energy it should deliver as much power as possible to the receiving-end apparatus, and (although efficiency is usually a secondary consideration) it should not make excessive demands on the sending-end equipment.

Matching of impedances is necessary for maximum power transfer. The general principle may be restated. If a transmission system is broken at any point, the impedance at that point looking toward the generator should be the conjugate of the impedance looking toward the load.

If the transmission system consists of generator, line, and load it is advantageous to avoid resonance on the line. For such "flat" operation of the line, both load and generator must be matched to the characteristic impedance of the line. If a proper impedance match is not provided at the load, there is resonance and standing waves of current and voltage appear on the line. If the impedance match at the generator is not correct, there is not maximum power transfer.

However, it is not essential for maximum power transfer that the line be "flat," without standing waves. If resonance is permitted on the line, the line itself may be used as an impedance transformer to match the load to the generator. Proper matching is achieved if the line is connected to the load, and the input impedance to the line is then the conjugate of the generator impedance. The impedance curves of Fig. 3-5 suggest that a wide range of input impedance is available if it is possible to adjust the length of a line with a mismatched load. It will be seen, however, that adjustment of length alone is not sufficient, for a change of length alters both magnitude and angle of the input impedance and it would be quite fortuitous if the magnitude and angle of the generator impedance were both matched by the same length of line.

It is an important and fairly obvious general principle that two variables cannot be independently adjusted if only one physical quantity can be changed; the number of physical adjustments must be at least equal to the number of quantities to be controlled.

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[9] King, Mimno, and Wing, "Transmission Lines, Antennas and Wave Guides" (Ref. 15).

However, if both the length of the line and also the characteristic impedance of the line can be varied, the line can be designed to match any load impedance to any generator impedance. A simple example is seen if the impedances of both load and generator are pure resistance; then the quarter-wavelength line mentioned in Sec. 10 will provide the required match if its characteristic impedance is the geometric mean of the two terminating resistances (Eq. 12-48).

Although a perfect match cannot be obtained by adjusting length only, it will in general be possible to go part of the way toward achieving maximum power transfer by this single adjustment. A sliding device known as a "trombone" or "line stretcher" is a simple means of varying the electrical length of a line even while it is in operation.

If a line is many wavelengths long, the undesirability of permitting standing waves on the line outweighs the simplicity of using the line itself for a matching transformer. Source and load must then be matched to the line. There are many ways of doing this. A section of line of different Z_0 may be inserted between the main line and the apparatus at either end, leaving the main line nonresonant. This has been mentioned. Or *stub lines*, open or short-circuited, may be connected to the main line, near the ends, for matching. Or a section of *tapered line*, with continuously varying characteristics, may be used to connect one part of the system to another of different impedance. There are many such devices. One or two of the more important will be discussed.

First, however, the *standing-wave ratio* of a line, and some of its implications, must be considered.

13. Standing-wave Ratio. The standing-wave ratio on a line is simply the ratio of the voltage at a point of maximum to the voltage at a point of minimum. It is important because it is an easily measured quantity that serves as an indication of whether or not the line is operating under favorable conditions.

Techniques of measurement that are useful at lower frequency cannot often be employed at frequencies higher than a thousand megacycles. One of the quantities that can be measured fairly easily and reliably at centimeter-wave frequencies is relative voltage. The insertion of a probe into a slot in the outer tube of a coaxial line permits determination of the relative electric field strength and hence the relative voltage at different points along the line. The details of rectification and amplification of the voltage picked up by such a probe need not be discussed here. Alternatively, a loop inserted into the magnetic field of the line will permit comparison of current at different points along the line. Either measurement will yield the standing-wave ratio on the line. It is particularly important that only the ratio of voltages (or currents) and not their absolute magnitudes are needed.

The symbol σ is frequently used for standing-wave ratio:

$$\sigma = \text{standing-wave ratio} = \frac{|V_{\text{max}}|}{|V_{\text{min}}|} \tag{12-60}$$

The standing-wave ratio is always positive, real, and not less than 1. The standing-wave ratio equals 1 on a *flat* or perfectly terminated line, on which there is no reflected traveling wave and hence no standing waves along the line. On an open or short-circuited line, with total reflection, so that the reflected wave is equal in magnitude to the incident wave, V_{min} is zero and σ is infinite.[10]

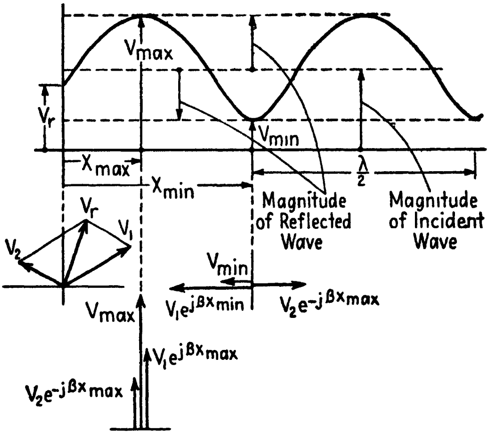

FIG. 12-8. Standing-wave ratio.

The standing-wave ratio is closely related to the ratio of reflected wave to incident wave ρ (and hence to the reflection factor), and to the impedance ratio ζ; it will be remembered that $\rho = V_2/V_1$ and $\zeta = Z_r/Z_0$ as in Eq. 2-30.

Figure 12-8 illustrates the relation between the standing-wave ratio and the traveling waves. It shows the voltage distribution along a line. The load voltage (at the left-hand side) is neither a maximum nor a minimum.

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[10] Equation 12-60 defines the *voltage* standing-wave ratio, which is the same as the *current* standing-wave ratio. The *power* standing-wave ratio η is the square of this:

$$\eta = \frac{|V_{\text{max}}|^2}{|V_{\text{min}}|^2} \tag{12-61}$$

A maximum is reached at a distance $x_{\max}$ along the line, and a minimum at $x_{\min}$. Corresponding vector diagrams are shown below. The incident and reflected waves are in phase at $x_{\max}$, and $V_{\max}$ is their arithmetic sum:[11]

$$| V_{\max} | = | V_1 | + | V_2 | \tag{12-62}$$

The incident and reflected waves are directly opposed at $x_{\min}$, so that

$$| V_{\min} | = | V_1 | - | V_2 | \tag{12-63}$$

Hence, by Eq. 12-60,

$$\sigma = \frac{| V_{\max} |}{| V_{\min} |} = \frac{| V_1 | + | V_2 |}{| V_1 | - | V_2 |} = \frac{1 + | V_2/V_1 |}{1 - | V_2/V_1 |} \tag{12-64}$$

Solving for the ratio of V_2 to V_1,

$$| \rho | = \left| \frac{V_2}{V_1} \right| = \frac{\sigma - 1}{\sigma + 1} = \frac{| V_{\max} | - | V_{\min} |}{| V_{\max} | + | V_{\min} |} \tag{12-65}$$

Since $V_{\max}$ and $V_{\min}$, or else their ratio σ, can readily be determined experimentally, this gives a means of finding the magnitude of the reflection coefficient. The reflection coefficient is in general complex, however, and its angle must also be determined. Let us refer again to Fig. 12-8.

The angle of the reflection factor is the angle between V_1 and V_2 at the load. For instance, in Fig. 12-8 it is 120 degrees. At the point of maximum voltage, where the two traveling-wave components are in phase, each has changed in phase by half this angle. Consequently

$$\text{Angle of reflection factor} = 2\beta x_{\max} \tag{12-66}$$

The angle of rotation of each vector, which is $\beta x_{\max}$, can also be written $2\pi(x_{\max}/\lambda)$. Either of these expressions gives the angle of rotation in radians, and if the angle is desired in degrees, we write 180 instead of π. Alternative forms for the reflection factor can be written in terms of the quarter-wavelength of the line as

$$\text{Angle of reflection factor} = \frac{x_{\max}}{\lambda/4}\,\pi \text{ radians} \tag{12-67}$$

or

$$\text{Angle of reflection factor} = \frac{x_{\max}}{\lambda/4}\,180 \text{ degrees} \tag{12-68}$$

---

[11] This discussion of standing-wave ratio is meaningful only on a line with negligible loss and negligible attenuation. In the absence of attenuation, the magnitudes $| V_1 |$ and $| V_2 |$ are constant all along the line. If this were not so, ρ would be variable with x, and σ would be meaningless, or at best some approximation would be necessary (see, for instance, Fig. 5-9).

If the wavelength on the line is not known, the half-wavelength is found experimentally as the distance between successive voltage minima, as indicated in Fig. 12-8.

As a practical matter, the voltage minimum is more sharply defined, and hence can be more accurately located than the maximum. That is, x_{min} can be measured more exactly than x_{max}. To take advantage of this, a slight change can be made in the above equations to put them in terms of x_{min}. Referring again to Fig. 12-8, a voltage minimum occurs where each of the component voltage vectors has rotated through an angle of $\beta x_{\text{max}} + \pi/2$. It is possible that if the line were differently terminated a voltage minimum would precede the first voltage maximum, in which case the first voltage minimum would occur where each of the component voltage vectors had rotated through an angle of $\beta x_{\text{max}} - \pi/2$. Allowing for either of these alternatives,

$$\beta x_{\text{min}} = \beta x_{\text{max}} \pm \frac{\pi}{2} \tag{12-69}$$

Substituting this into Eqs. 12-66 to 12-68:

$$\text{Angle of reflection factor} = 2\beta x_{\text{min}} \pm \pi \text{ radians} \tag{12-70}$$

$$= \frac{x_{\text{min}}}{\lambda/4}\pi \pm \pi \text{ radians} \tag{12-71}$$

$$= \frac{x_{\text{min}}}{\lambda/4} 180 \pm 180 \text{ degrees} \tag{12-72}$$

Either the plus or minus sign can be used in these equations as is more convenient; neither is wrong.

Having now a relation between the standing-wave ratio, the position of voltage minimum (which are experimental data), and the reflection factor, another step will relate also the terminal impedance. From Eq. 2-30,

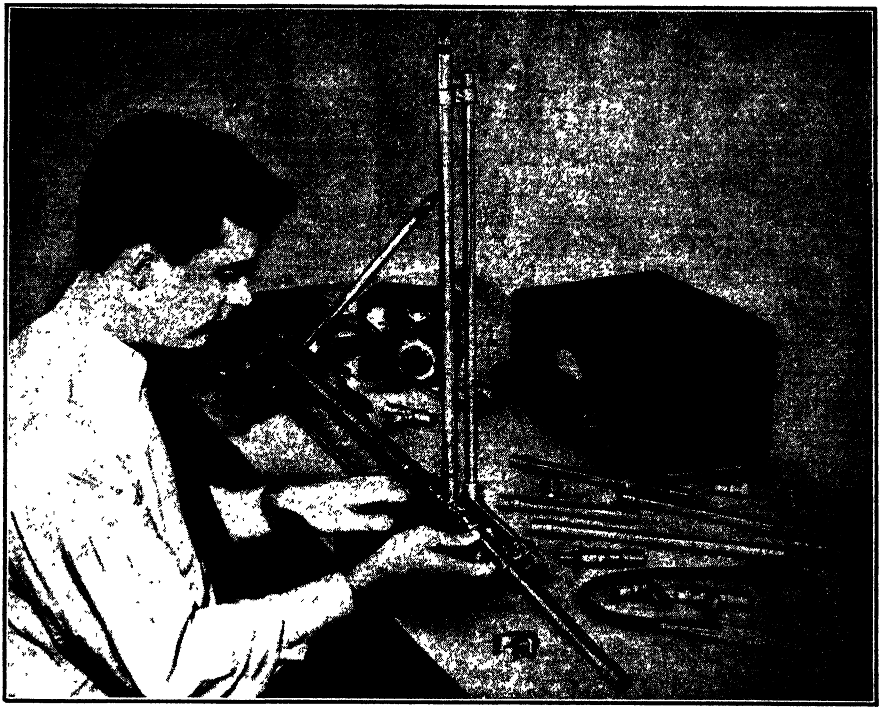
$$\rho = \frac{V_2}{V_1} = \frac{Z_r/Z_0 - 1}{Z_r/Z_0 + 1} = \frac{\zeta - 1}{\zeta + 1} \tag{12-73}$$

Solution for impedance gives

$$\zeta = \frac{Z_r}{Z_0} = \frac{1 + V_2/V_1}{1 - V_2/V_1} = \frac{1 + \rho}{1 - \rho} \tag{12-74}$$

Hence, if the standing-wave ratio is known, together with the location on the line of a voltage maximum or minimum, the load impedance can be found. It is assumed that Z_0 is known or can easily be computed.

This gives an important and valuable means of measuring impedance at high frequency. The method is to connect an element of unknown



A slotted coaxial line assembly for measurement of standing waves, 300 to 5,000 megacycles. A length of rigid air-insulated line is being attached to the right-hand end of the slotted line; a section of "patch cord" (flexible coaxial line with polyethylene insulation) is connected at the left. A tuning stub is upright at the end of the slotted line, and a shorter one projects from the movable head that carries a probe along the slot. The probe is connected to an amplifier and voltmeter in the background. Various line fittings are shown. All lines and attachments have 50-ohm characteristic impedance. (*Courtesy of General Radio Company.*)

impedance onto the receiving end of a slotted transmission line that has a movable probe. The relative values of $V_{\max}$ and $V_{\min}$ are recorded, and also the distances from the receiving-end terminals to the first and second voltage minima. The reflection factor ρ is then computed from Eqs. 12-65 and 12-72, and Z_r is computed from 12-74. Further discussion is given in Sec. 16.

If it is known that Z_r is *purely resistive*, we have the simple relation that either

$$\frac{Z_r}{Z_0} = \frac{V_{\max}}{V_{\min}} = \sigma \qquad \text{or} \qquad \frac{Z_r}{Z_0} = \frac{V_{\min}}{V_{\max}} = \frac{1}{\sigma} \tag{12-75}$$

If V_2/V_1 is real and positive, the first of these equations is correct, as is seen by comparing Eq. 12-74 and Eq. 12-64, letting $V_2/V_1 = |V_2/V_1|$.

The second equation is correct if $V_2/V_1 = -|V_2/V_1|$. It will be seen that the first equation applies if Z_r is greater than Z_0 and the second if Z_r is less than Z_0. The former applies if the load is at a point of voltage maximum on the line, the latter if at a minimum.

The standing-wave ratio on a line is often a matter of interest in determining proper operation of the terminal equipment. It may be specified that an antenna, for instance, should operate on the end of a certain line without producing a standing-wave ratio on the line in excess of a given amount. For making such tests the obvious methods of measurement, using a slotted line as suggested above, are sometimes too slow, and a direct indicating device is desired. The *directional coupler* makes such a device possible.

Consider that the main transmission line shown in Fig. 12-9 is carrying energy in a wave traveling from left to right. Two connections are made from this main line to a second line running parallel; each connection transfers only a very small part of the energy from the main line to the secondary line. Energy passing through the first connection will start waves traveling in both directions along the secondary line, starting at *a*. Energy passing through the second connection will start waves going both ways from *b*. If the two connections are identical and the distance *ab* between them is $\frac{1}{4}$ wavelength, the wave traveling toward the right from *a* will reinforce the wave traveling toward the right from *b*. But the wave traveling left from *a* will exactly cancel the wave traveling left from *b*; the wave from *b* will be equal and out of phase because of having traveled just $\frac{1}{2}$ wavelength farther. Hence, a wave traveling from left to right in the main line will produce a wave traveling from left to right in the secondary line, but will not produce a wave traveling from right to left in the secondary line. The secondary line is terminated at both *c* and *d* with its characteristic impedance, to prevent reflections, and meters at *c* and *d* indicate the arrival of energy. A wave traveling from left to right in the main line will be measured at *d*, but not at *c*.

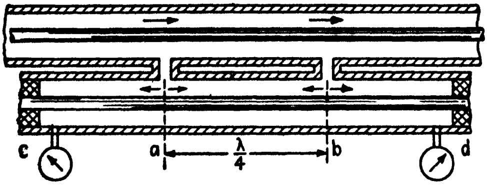


FIG. 12-9. A type of directional coupler.

Since the directional coupler is symmetrical, a wave traveling from right to left in the main line will be measured at *c* but not at *d*. The ratio of the reading at *c* to that at *d* can therefore be the ratio of the incident wave to the reflected wave in the main line. If the main line is *flat*, one instrument will indicate nothing.

It is essential that the connections in the directional coupler be identical and that little energy be removed from the main line. The nature of the connection is not otherwise important, and it is found that a mere hole from one line into the other gives satisfactory results, permitting the electromagnetic field to pass through.[12]

B. IMPEDANCE CHARTS

14. The Rectangular Impedance Chart. Input impedance is so important in the use of transmission lines at radio frequencies that several impedance charts have been devised and published, and have found wide use. The equations of impedance are complicated enough to make the use of a chart worth while.

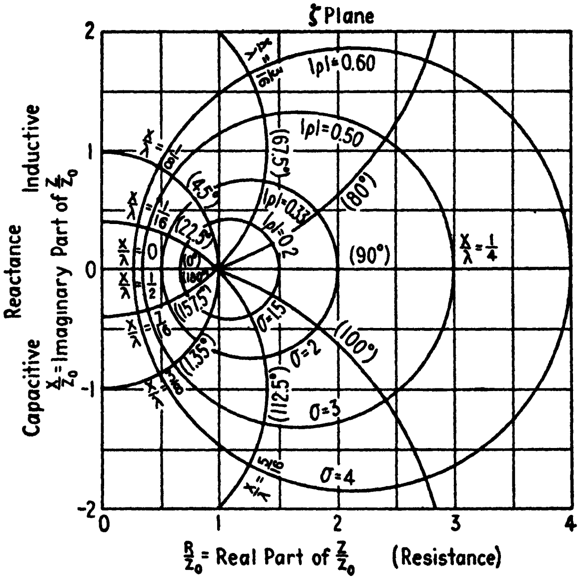


FIG. 12-10. The rectangular impedance chart.

One type of chart is shown in Fig. 12-10. The coordinates of the chart are the real and imaginary components of Z/Z_0, which may be called

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[12] There is another type of directional coupler, the operation of which depends on the angle between the main line and the secondary line, only one hole being used. This and other ingenious measuring devices are described in books on techniques, as Refs. 22 and 24.

R/Z_0 and X/Z_0, respectively. Thus the chart lies in the complex ζ plane, which was introduced in Fig. 2-9*a*. The circles that enclose the point $Z/Z_0 = 1$ give both the standing-wave ratio σ and the magnitude of ρ. The orthogonal set of circles, all of which pass through the point $Z/Z_0 = 1$, give distance along the line measured as fractions of a wavelength. Use of the chart will be considered first, and then the derivation will be given.

As used in this chart, ρ is the ratio of the voltage of the reflected wave to the voltage of the incident wave at any given point on the line; Z is the impedance, or ratio of voltage to current, at that same point. (In an earlier chapter, ρ represented the same ratio, but applied only at the receiving end of the line, and Z was the impedance at the receiving end, Z_r. We now generalize to give ρ and Z meaning at all points of the line.)

As *an example*, consider a line that is $\frac{5}{16}$ wavelength long and is terminated in a purely resistive load that is half the characteristic impedance. For such a load, $Z_r/Z_0 = \frac{1}{2}$. Locate this point in the chart; it lies on the horizontal axis one square from the left edge. Note that the circle $|\rho| = 0.33$ passes through this point.

Now, if the line under consideration may be considered to be without loss, traveling waves on the line are not attenuated, the magnitude of the incident wave is the same everywhere on the line, the magnitude of the reflected wave is the same everywhere on the line, and hence the magnitude of their ratio is the same everywhere on the line. This means that $|\rho|$ is constant along the line. To find Z at any point along the line, we follow the circle of constant $|\rho|$, which in this example is the circle $|\rho| = 0.33$; moving in a clockwise direction around the circle corresponds to moving away from the load and toward the generator on the actual line.

Let us find the sending-end impedance of this line for which the length x is given in terms of the wavelength by stating that $x/\lambda = \frac{5}{16}$. Our initial point, at the load, lies on the line of the chart marked $x/\lambda = 0$. Our final point, at the sending end, therefore lies on the circle marked $x/\lambda = \frac{5}{16}$, and we follow around the circle marked $|\rho| = 0.33$ until it intersects the circle marked $x/\lambda = \frac{5}{16}$. At this point of intersection, we read Z/Z_0 from the coordinates, getting a value of about $1.39 - j0.73$. (It is hardly necessary to mention that this numerical value was obtained from a chart with finer divisions than Fig. 12-10.) The sending-end impedance is then found by multiplying Z_0 by this complex number. The result is an input impedance with resistive and capacitive components, both fairly large as compared to the load resistance.

The impedance for any other length of line can be found by following the circle for $|\rho| = 0.33$ until it intersects the appropriate circle for x/λ. If the length of the transmission line is more than a half-wavelength, we must go around the circle more than one complete revolution. Thus if the line were $\frac{3}{4}$ wavelength long, the sending-end point would be found by going once around the circle for the first half-wavelength and then continuing a distance corresponding to an additional $\frac{1}{4}$ wavelength. The chart reading at such a point is $Z/Z_0 = 2 + j0$, from which the input impedance is twice the characteristic impedance.

If the load impedance is not purely resistive, the procedure is the same but one extra step is required. As *another example*, consider the same line with a load such that $Z_r/Z_0 = 0.6 + j0.8$. The point of the chart corresponding to this impedance lies on the circle $|\rho| = 0.50$. It also lies on the intersecting circle for $x/\lambda = \frac{1}{8}$. The calibration of these distance circles, however, is only relative. We know that the point just located is truly the receiving-end impedance, where the true value of x/λ is zero. To find impedance at points along the line, we move around the circle for $|\rho| = 0.50$ in a clockwise direction, going the appropriate distance as measured *from this starting point.* Hence, if the line is $\frac{5}{16}$ wavelength long, we go around the circle $\frac{5}{16}$ wavelength from our starting point (which happens to be on the curve marked $\frac{1}{8}$ wavelength); thus we go to the intersection with the curve for $x/\lambda = \frac{5}{16} + \frac{1}{8}$ or $\frac{7}{16}$. The impedance read at this point is $Z/Z_0 = 0.38 - j0.36$, and this is the input impedance to the $\frac{5}{16}$-wavelength line.

As a matter of practical convenience, such charts commonly have the circles marked with values of standing-wave ratio σ instead of $|\rho|$. In Fig. 12-10, both values are given on each circle. It is easy to convert from one to the other by Eq. 12-65:

$$|\rho| = \frac{\sigma - 1}{\sigma + 1} \qquad \text{or} \qquad \sigma = \frac{1 + |\rho|}{1 - |\rho|} \tag{12-76}$$

The standing-wave ratio σ is more often of interest in practical work than is $|\rho|$, the magnitude of the ratio of reflected voltage to incident voltage.

It will be seen that each circle for a given value of standing-wave ratio σ is tangent at the horizontal axis to a vertical line with the identical value of Z/Z_0. Thus the circle for $\sigma = 3$ is tangent to the vertical line for $R/Z_0 = 3$. This follows from Eq. 12-75; each constant-σ circle intersects the axis at the two points σ and $1/\sigma$.

The circles of the orthogonal family are commonly marked in terms of distance along the line, either in fractions of a wavelength, or in degrees with the convention that 360 degrees equals 1 wavelength. Both of these scales are given in Fig. 12-10. Mathematically, these are more properly circles of constant *angle of* ρ, as the first set of circles were for constant values of *magnitude of* ρ. However, angle of ρ increases in direct proportion to distance along the transmission line (see Eq. 12-66), and it is permissible to calibrate the curves in Fig. 12-10 in terms of x, which may be measured in any way that is practically convenient.[13]

The line marked $x/\lambda = 0$ does not correspond, in fact, to the actual end of the line (as was seen in the second example above) unless the load is purely resistive and less than Z_0 in magnitude. However, some scale is needed and this is the usual arbitrary choice of zero. In using the chart, distance along the line is measured from the starting point, as was illustrated in the example, and the figures given on the circles are relative distance.

The *derivation of this chart* is from Eq. 12-73:

$$\rho = \frac{\zeta - 1}{\zeta + 1} \tag{12-77}$$

Curves giving values of magnitude of ρ, and curves giving values of angle of ρ, are plotted on Cartesian coordinates that give the real and imaginary components of ζ. These curves turn out to be orthogonal families of circles.[14]

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[13] Note that the angular values given in Fig. 12-10, and commonly given in all such charts, are a measure of distance along the transmission line, and are half the corresponding value of the angle of ρ. That is, for $\frac{1}{2}$ wavelength $\beta x = 180$ degrees; whereas the angle of ρ is increased by 360 degrees along a half-wavelength line. This accounts for the anomaly that one revolution around a circle in Fig. 12-10 is marked 180 degrees; this is the value of βx, not the angle of ρ.

[14] A mathematical development of Fig. 12-10 follows:

$$\rho = \frac{\zeta - 1}{\zeta + 1} \tag{12-77}$$

Let

$$\rho = re^{j\theta} \tag{12-78}$$

and

$$\zeta = x + jy \tag{12-79}$$

Then

$$r = \left| \frac{x + jy - 1}{x + jy + 1} \right| \tag{12-80}$$

$$r^2 = \frac{(x - 1)^2 + y^2}{(x + 1)^2 + y^2} \tag{12-81}$$

from which

$$\left(x - \frac{1 + r^2}{1 - r^2}\right)^2 + y^2 = \frac{4r^2}{(1 - r^2)^2} \tag{12-82}$$

For constant r, r being the magnitude of ρ, this is the equation of a circle in the ζ plane. The center of the circle is at $x = (1 + r^2)/(1 - r^2)$, $y = 0$; the radius is

15. Other Impedance Charts. In considering impedance charts, it is natural to inquire why charts are not made from the familiar impedance Eq. 5-32, or Eq. 5-37, derived from it, which is repeated here:

$$\frac{Z}{Z_0} = j \tan (Bx + D) \tag{12-87}$$

The answer is that the chart of Fig. 12-10 is indeed a graphic presentation of this Eq. 12-87 if rightly interpreted.

The most obvious way to chart Eq. 12-87 would be to plot Z/Z_0 in a plane in which the coordinates are the real and imaginary components of $(Bx + D)$. This is essentially a plot of the complex tangent function, which may be found in appropriate references.[15] The disadvantage of such a chart is that one follows straight lines (instead of circles) in tracing the values of impedance along a lossless transmission line and, if the line is long, one runs off the chart.

An alternative chart can be obtained from Eq. 12-87 by plotting curves of Bx in the ζ plane; this is the inverse of the chart just proposed which plots ζ in the Bx plane. When this is done, Fig. 12-10 again results.[16]

The use of the chart is, of course, the same whether it is considered to be derived from Eq. 12-87 or from Eq. 12-77.

$2r/(1 - r^2)$. Plotting for various values of r gives the family of circles enclosing the point $x = 1, y = 0$.

$$\rho = \frac{x - 1 + jy}{x + 1 + jy} \tag{12-83}$$

$$\theta = \tan^{-1} \frac{y}{x - 1} - \tan^{-1} \frac{y}{x + 1} \tag{12-84}$$

Let

$$m = \tan \theta = \frac{\dfrac{y}{x - 1} - \dfrac{y}{x + 1}}{1 + \dfrac{y}{x - 1}\dfrac{y}{x + 1}} = \frac{2y}{x^2 + y^2 - 1} \tag{12-85}$$

from which

$$x^2 + \left(y - \frac{1}{m}\right)^2 = 1 + \frac{1}{m^2} \tag{12-86}$$

For constant m, m being the tangent of the angle of ρ, this is the equation of a circle in the ζ plane. The center of each circle is at $x = 0$, $y = 1/m$; and the radius is $\sqrt{1 + 1/m^2}$. Plotting for various values of m gives the family of circles passing through the point $x = 1, y = 0$.

[15] For instance, Jahnke and Emde, "Tables of Functions with Formulae and Curves" (Ref. 11).

[16] See J. C. Slater, "Microwave Transmission," p. 30 (Ref. 32).

16. The Smith Chart. The impedance chart that probably finds most frequent use in practice is shown in Fig. 12-11. It is commonly known as the Smith chart, and is closely related to the rectangular chart of Fig. 12-10.

The chart of Fig. 12-10 is basically a plot of ρ in the ζ plane, using Eq. 12-77. The same equation may take the form

$$\zeta = \frac{1+\rho}{1-\rho} \tag{12-88}$$

and we may plot ζ in the ρ plane.[17]

More specifically, the real and imaginary components of impedance, R/Z_0 and X/Z_0, are plotted on a system of coordinates representing ρ. (The ρ plane was discussed in connection with Fig. 2-9*b*.)

In Fig. 12-11, the origin of coordinates is at the center of the chart. It

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[17] The mathematical derivation of the Smith chart follows. (In the phraseology of complex variable theory, it is a conformal circle-preserving transformation of Fig. 12-10.)

$$\zeta = \frac{1+\rho}{1-\rho} \tag{12-88}$$

Let

$$\zeta = r + jx \qquad \rho = u + jv \tag{12-89}$$

$$r + jx = \frac{1+u+jv}{1-u-jv} = \frac{1-u^2-v^2+j2v}{(1-u)^2+v^2} \tag{12-90}$$

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad x = \frac{2v}{(1-u)^2+v^2} \tag{12-91}$$

from which

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2} \tag{12-92}$$

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2} \tag{12-93}$$

For constant r, Eq. 12-92 is the equation of a circle in the ρ plane; r is the real (resistance) component of ζ. The center of each circle is at $u = r/(1+r)$, $v = 0$, and u is between 0 and 1 for positive values of r. Since u and v are the coordinates of the chart, and since only positive values of r are physically possible, the centers of circles are on the horizontal axis between the origin and the point $+1$. The radius of each circle is $1/(1+r)$.

For constant x, Eq. 12-93 is the equation of a circle in the ρ plane; x is the imaginary (reactance) component of ζ. The center of each circle is at $u = 1$, $v = 1/x$. The centers of this family of circles fall on a vertical line one unit to the right of the origin; above the horizontal axis if x is positive (inductive) and below if negative (capacitive). The radius of each circle is $1/x$.

is customary in preparing the Smith chart[18] to omit the cross-sectional lines that would indicate ρ, leaving it to the user to draw in whatever such lines may be needed. Since ρ is commonly used as a magnitude and an angle (rather than being expressed as the sum of real and imaginary components), polar coordinates for ρ are convenient. One circle and three radial lines have been drawn in Fig. 12-11 to illustrate the appearance of circles of constant magnitude of ρ and the radial lines of constant angle of ρ.

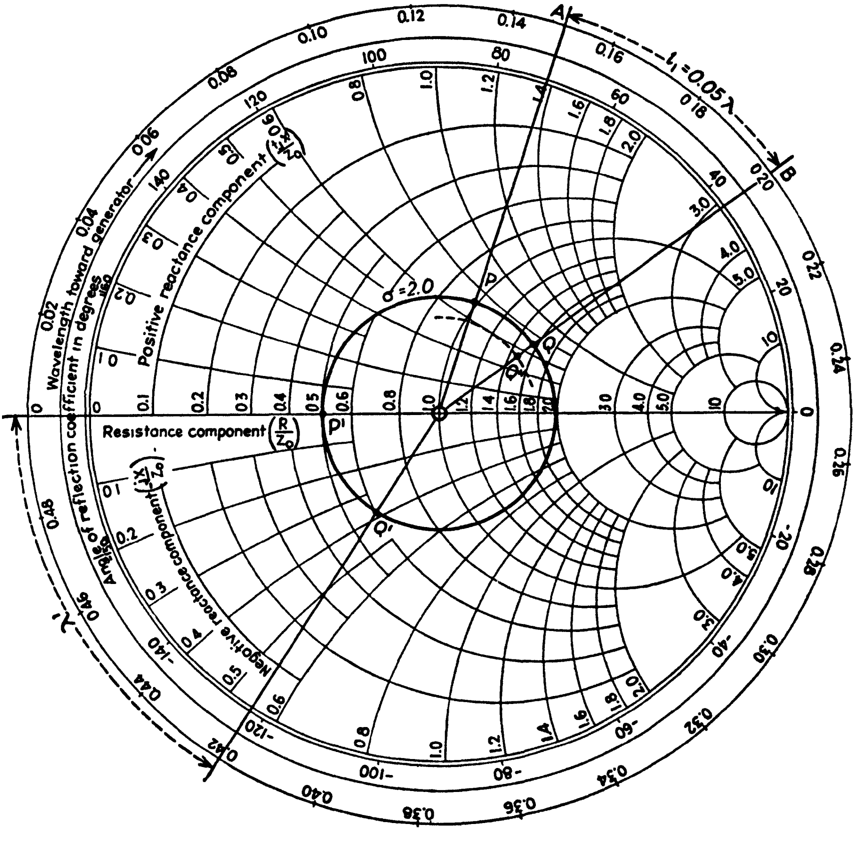

FIG. 12-11. The Smith chart. (*From F. E. Terman, "Radio Engineering," McGraw-Hill Book Company, Inc., New York*, 1947.)

The following comments on the Smith chart are similar to those that were made relative to Fig. 12-10. Radial lines are lines of constant angle of ρ, or of constant βx, but most usefully they are lines of constant x/λ and are so labeled on the chart. The outermost circular scale is calibrated in distance along the transmission line, measured in fractions of a wavelength. The line OA, for example, is the radial line for which x is 0.15 wavelength. This distance must be measured from some specified reference point, and in the Smith chart the distance scale begins at zero at the extreme left; this results in distance being measured (as it was in Fig. 12-10) from a point of voltage minimum on the transmission line.

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[18] The Smith chart is published on printed forms of various sizes. These may be purchased cheaply, much like cross-section paper. A circular slide rule in Smith chart form may also be obtained. See P. H. Smith, Transmission Line Calculator, *Electronics*, January, 1939, and January, 1944.

A circle about the center of the Smith chart is a circle of constant $|\rho|$, but more usefully it may be interpreted as a circle of constant standing-wave ratio. The circle drawn in Fig. 12-11 is for a standing-wave ratio of 2. No scale of standing-wave ratio is given on the Smith chart, so the circle is identified by noting the point at which it crosses the horizontal axis. The standing-wave ratio is equal to the impedance ratio R/Z_0 at the point of intersection with the right-hand half of the horizontal axis. The standing-wave-ratio circles cross the right-hand half of the axis at $R/Z_0 = \sigma$ and the left-hand half at $R/Z_0 = 1/\sigma$. (See the explanation following Eq. 12-76.)

The locus of points along a lossless transmission line is a circle of constant σ on the Smith chart, as it is in Fig. 12-10. The locus of points along a line with loss is a spiral closing in on the center of the chart. The chart is mainly used for work with practically lossless lines, and a few examples will illustrate its application.

What is the impedance at a point of voltage minimum if the standing-wave ratio is 2? This may be read from the chart at P' where the circle for $\sigma = 2$ crosses the left-hand half of the horizontal axis. It is $R/Z_0 = 0.5$, $X/Z_0 = 0$. (This answer was already known, but provides an elementary illustration of the use of the chart.) See Fig. 12-12.

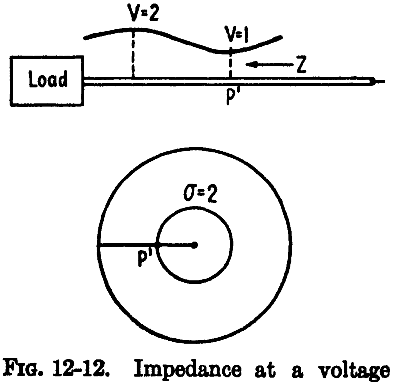

FIG. 12-12. Impedance at a voltage minimum.

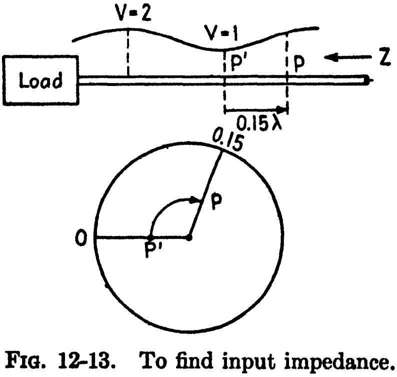

FIG. 12-13. To find input impedance.

What is the impedance at a point that is 0.15 *of a wavelength toward the generator from a point of voltage minimum if the standing-wave ratio is* 2? Enter the chart at P' where the circle $\sigma = 2$ crosses the left-hand half of the horizontal axis; this is the point of voltage minimum. Move around that circle in a clockwise direction (toward the generator) through an angle that may be measured on the outermost circular scale as 0.15 wavelength (to the radial line OA), reaching the point P. Reading impedance at this point, $Z = Z_0(0.98 + j0.70)$ (see Fig. 12-13).

A line is terminated in a load such that $Z_r/Z_0 = 0.98 + j0.70$. *The line is* 0.05 *wavelength long. What is the input impedance?* Enter the chart with the given load impedance, which locates the point P. Draw a circle through this point, with center at the origin, and measure an angle equivalent to 0.05 wavelength in a clockwise direction, using the outermost scale. At Q, so determined, read the input impedance $Z = Z_0(1.57 + j0.7)$.

The standing-wave ratio on a line is 2.0, *and there is a voltage minimum* 0.08 *wavelength from the load* (see Fig. 12-14). *What is the load impedance?* Enter the chart at the point at which the circle for $\sigma = 2.0$ crosses the left-hand half of the horizontal axis; this point P' corresponds to the observed point of minimum voltage. Measure along this circle an angle of 0.08 wavelength, using the outermost scale. Measure this angle in a counterclockwise direction, corresponding to distance *from* the generator, for the distance that is being measured is from the minimum-voltage point to the load. At Q' read the load impedance, $Z = Z_0(0.6 - j0.38)$ (see also Fig. 12-15).

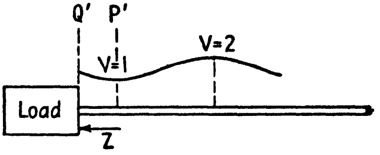

FIG. 12-14. Line with unknown load.

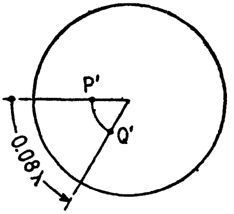

FIG. 12-15. To find load impedance.

What is the reflection factor at such a load? The magnitude, by Eq. 12-65, is $(\sigma - 1)/(\sigma + 1) = 0.33$. The angle is read from the next outermost scale of the Smith chart as -122 degrees (or, from Eq. 12-72, it is -122.4 degrees).

Using the same data, except that the minimum-voltage point is 0.58 *wavelength from the load, what is the load impedance?* The same. Impedance repeats each half-wavelength. If the transmission line is longer than $\frac{1}{2}$ wavelength, the point is carried around the Smith chart the appropriate number of complete revolutions. This makes no difference in the answer unless there is loss along the line.

As a typical measurements problem, see Fig. 12-16. *An antenna is fed through a slotted line. The line is first short-circuited at the antenna and a voltage minimum is observed in the slotted line. The short circuit is then removed, and energy goes to the antenna; the point of minimum voltage in the slotted line moves* 0.08 *wavelength toward the generator and the standing-wave ratio becomes* 2.0. *Find the antenna impedance.* During short circuit there was necessarily a voltage minimum at the receiving end of the line. With the short-circuit removed, it is known that a voltage minimum must occur 0.08 wavelength from the receiving end (although this particular minimum cannot be observed), which means that the antenna is 0.08 wavelength beyond a voltage minimum. On the Smith chart, the voltage minimum is at point P'; the antenna impedance is read at point Q', 0.08 wavelength farther from the generator. P' and Q' are both on the circle for a standing-wave ratio of 2.0. The impedance at Q' is the antenna impedance and is read from the chart as $Z_0(0.6 - j0.38)$. See Fig. 12-15 for this problem also.

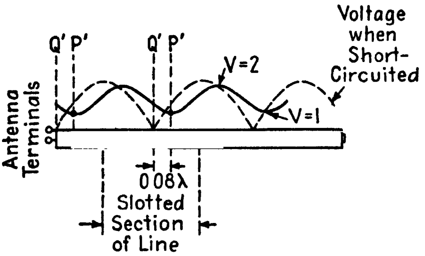

FIG. 12-16. A typical impedance-measurement problem.

To Include Loss. If a transmission line has significant loss, the value of αx as well as the value of βx must be used in the chart. Radial distance on the Smith chart is the magnitude of ρ, and the magnitude of ρ depends on attenuation as follows:[19]

$$\frac{|\rho| \text{ at point } p}{|\rho| \text{ at point } q} = e^{-2\alpha x} \tag{12-94}$$

where p and q are any two points on the line, the latter being closer to the generator end by the distance x.

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[19] The distance from point p to point q is x. Hence, if the voltage of the incident wave at point p is V_{1p}, the voltage of the incident wave at point q is $V_{1q} = V_{1p}e^{jBx}$. Similarly, the voltage of the reflected wave at point q in terms of the voltage at point p is $V_{2q} = V_{2p}e^{-jBx}$. Therefore, if

$$\rho_q = \frac{V_{2q}}{V_{1q}} = \frac{V_{2p}e^{-jBx}}{V_{1p}e^{jBx}} = \frac{V_{2p}}{V_{1p}} e^{-2jBx} = \rho_p e^{-2jBx} \tag{12-95}$$

it follows that

$$|\rho_q| = |\rho_p| e^{-2\alpha x} \tag{12-96}$$

$$\text{Angle of } \rho_q = \text{angle of } \rho_p - 2\beta x \tag{12-97}$$

To use this relation in the Smith chart, it is necessary to consider that the locus of impedance values along a line with loss is not a circle (as for a lossless line) but a spiral. As βx increases, at points along the line more distant from the load, one moves around a circle on the chart, and as αx simultaneously increases, one moves inward toward the center of the chart. In actual computation, $e^{-2\alpha x}$ is determined from a table (or chart) of exponentials; a preliminary point on the Smith chart is then found for the correct value of βx but neglecting attenuation, and finally a revised point is found on the chart at a radius less than that of the preliminary point by the factor $e^{-2\alpha x}$. An example will show the method more clearly than further explanation.

A line is 3.70 *wavelengths long, and the loss in this length is* 1.14 *db. The line is terminated in a pure resistance load, giving a voltage minimum at the load. The standing-wave ratio on the line is* 2.0. *What is the sending-end impedance?* Enter the chart at P', the receiving-end point. Travel in a clockwise direction seven times around the circle (this takes care of 3.50 wavelengths) plus 0.20 wavelength as measured on the outer scale. This locates the line OB. If the transmission line were without loss, the sending-end point would be at Q, for βx has been taken into account by going around the circle. Since loss is to be considered, αx must now be taken into account by a change of radius. The attenuation, given in decibels, is changed to nepers:

$$\alpha x = \frac{1.14}{8.686} = 0.131 \text{ nepers}$$

From a table, $e^{-2\alpha x} = 0.77$; this is the factor by which the radius on the Smith chart is to be decreased. Locate[20] point Q'' on the line OB at a radius that is 0.77 times the radius OQ, and read impedance at Q''. It is $Z/Z_0 = 1.48 + j0.53$.

17. The Smith Chart as an Admittance Diagram. As seen in these examples, the Smith chart can often be useful for computing impedance. It is perhaps even more often used for computing admittance, as many problems that involve parallel connection of circuits are more easily solved that way.

It is remarkable that the Smith chart can be used without any alterations as an admittance chart, although it was constructed as an impedance chart. The manipulation for all problems involving admittance is identically the same as for impedances. The families of circles on the chart give values of conductance (instead of resistance) and susceptance (instead of reactance). Distance toward the generator is read the same, and so is standing-wave ratio. Magnitude of the reflection factor is unchanged.

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[20] A circular slide rule embodying the Smith chart is produced with a rotating radial arm having scales to take attenuation into account (see Refs. 22 and 33).

The input admittance, or the admittance at any point, is of course the reciprocal of the impedance at that point, and the characteristic admittance is the reciprocal of the characteristic impedance.[21] The point at the center of the chart, $Y/Y_0 = 1$, represents a line terminated in its characteristic admittance. Points on the left half of the horizontal axis represent low conductive admittance, and the extreme left-hand point is zero admittance or open circuit. (This contrasts with its correspondence to short circuit when the chart is used for impedances.) Similarly, the right-hand point on the admittance chart is infinite admittance or short circuit.

Susceptances in the upper half of the chart are capacitive.[22] Thus an open-circuited line that is less than a quarter-wavelength long will be seen to have capacitive susceptance, as we know it must. Susceptances in the

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[21] Thus

$$Y_0 = \frac{1}{Z_0} \tag{12-98}$$

and at points p and q

$$Y_p = \frac{1}{Z_p} \qquad Y_q = \frac{1}{Z_q} \tag{12-99}$$

From Eqs. 12-73 and 12-95, ρ_q may be written equal to

$$\frac{\dfrac{Z_q}{Z_0} - 1}{\dfrac{Z_q}{Z_0} + 1} = \frac{\dfrac{Z_p}{Z_0} - 1}{\dfrac{Z_p}{Z_0} + 1} e^{-2jBx} \tag{12-100}$$

Substitution of Eqs. 12-98 and 12-99 gives

$$\frac{\dfrac{Y_q}{Y_0} - 1}{\dfrac{Y_q}{Y_0} + 1} = \frac{\dfrac{Y_p}{Y_0} - 1}{\dfrac{Y_p}{Y_0} + 1} e^{-2jBx} \tag{12-101}$$

Since this equation in admittance is identical in form with Eq. 12-100 in impedance, and since either equation represents the process of entering a chart with the impedance (or admittance) at point p to locate ρ at p, then moving on the chart through Bx to locate point q, and there reading the impedance (or admittance) at point q, it is apparent that the use of the chart in one case is the same as the use of the chart in the other.

However, the fractions of Eq. 12-100 express ρ_p and ρ_q, whereas those of Eq. 12-101 express $-\rho_p$ and $-\rho_q$. Hence angles of reflection coefficient printed in Fig. 12-11 must be changed by 180 degrees if this scale is used for admittance. Very few uses of the chart involve reading this scale.

[22] Authors are not in agreement as to whether capacitive susceptance is to be called positive or negative. Best practice appears to favor the former. Inductive susceptance is then negative.

lower half of the chart are inductive. Use of the chart as an admittance diagram is illustrated in the next section.

18. Impedance Matching by Stub Line. Impedance matching by means of reactive networks was discussed in Sec. 7, Chap. 9. Lines for centimeter waves cannot be matched by using networks of inductance coils and condensers, but the same general principle can be applied by using appropriate lengths of short-circuited lines to serve as reactive elements. It will be remembered that the input impedance to a short-circuited lossless line can have any purely reactive value. Hence a short-circuited line can be, at its input terminals, equivalent to either an inductance or a capacitance of any desired size.

A coaxial line is shown in Fig. 12-17*a* connected to a load; line and load impedances are not matched. If a short-circuited line of proper length, called a tuning stub, is connected to the main line at the right distance from the load, a correct match will be provided at the point of connection of the stub. There will be resonance on the line from load to stub, and on the stub line (these take the place of the two reactive elements of Sec. 7, Chap. 9) but the main line will operate *flat.*

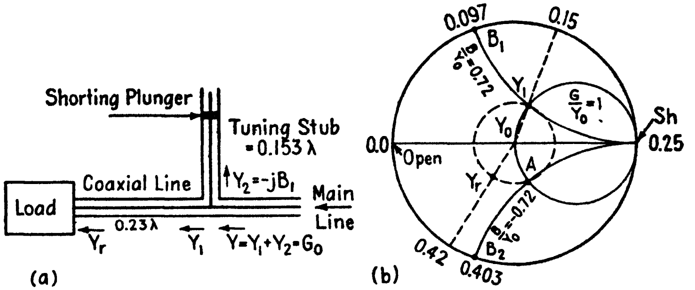

FIG. 12-17. Use of Smith chart for single-stub impedance matching.

The Smith chart gives the necessary information about the single-stub tuner, as it is called. An example will show the method, and will at the same time illustrate the use of the Smith chart as an admittance diagram.

A load has admittance Y_r *such that* $Y_r/Y_0 = 0.6 - j0.38$. *Find the length and location for a short-circuited tuning stub.* Referring to Fig. 12-17*b*, load admittance is represented at Y_r. Admittances along the coaxial line correspond to points along the dash-line circle from Y_r to Y_1.

Y_1 is the point of intersection with the constant-conductance circle for which $G/Y_0 = 1$. This is a unique circle, for all admittance points on this circle have conductance equal to 1, although the susceptance may be anything. At the point Y_1, for example, the conductance is 1 and the susceptance is 0.70. The susceptance can be tuned out by connecting an equal and opposite susceptance at this point. There then remains an admittance of merely $Y/Y_0 = 1$, and this is exactly what is needed to provide an impedance match. Manipulation of the chart is therefore as follows.

From Y_r, follow the dash circle clockwise to intersect the $G/Y_0 = 1$ circle at a point we may call Y_1. Distance toward generator is read from the outermost scale; from Y_r to Y_1 it is, in this example, 0.23 wavelength. This gives the location of the stub. At Y_1, susceptance is $B_1/Y_0 = 0.70$; this must be tuned out by a stub line with susceptance $B_2/Y_0 = -0.70$. This locates the point B_2 on the outer circle, which must be the input impedance to a short-circuited stub line. The length of the stub line is the distance from B_2 to the point *Sh*, the point on the chart corresponding to short circuit, and in this example it is 0.153 wavelength. Location and length of the stub have thus been determined.

Note that the circle Y_rY_1 intersects the circle $G/Y_0 = 1$ in two points, Y_1 and A. An impedance match can be achieved by attaching a stub at either of these points. A short-circuited stub connected at A, however, would have to be more than a quarter-wavelength long; it would have to have capacitive input susceptance to compensate for the inductive susceptance of the main line at this point. It is advantageous to use the shortest possible stub line, to keep losses small. Also, the resonant section of the main line should preferably be short. Hence it is usually not desirable to connect the tuning stub at A.

Yet another alternative is to connect an open-circuited tuning stub at A; an open-circuited stub would be less than $\frac{1}{4}$ wavelength long. Open stubs have two disadvantages: they radiate energy from the open end, and they are not easily adjusted as to length. Despite the most careful design, final tuning must always be done by observation of standing-wave ratio after physical assembly of the parts. The length of a short-circuited line is rather easily adjusted with a movable plunger (see Sec. 10), but the length of an open line can be changed only by means of rather complicated sliding joints. Short-circuited lines are ordinarily preferred for tuning.

The disadvantage of the single-stub tuner is that the position of the stub, as well as its length, must be adjustable. This is inconvenient, and double-stub or triple-stub tuners are often preferable.

19. The Double-stub Tuner. An impedance match can be obtained in many (although not all) cases by using two short-circuited stub lines of fixed position but variable length as shown in Fig. 12-18. The stub lines are shown $\frac{1}{4}$ wavelength apart; this is a practical design, although other spacings are sometimes preferable. The two stubs and the section of line between them will be called the tuning section. Admittance looking toward the load at the output of the tuning section is Y_3, and this point is located on the Smith chart. It is a purely arbitrary value, depending on the load (but we must exclude values of Y_3 that would fall within the circle $G/Y_0 = 1$, which is marked Circle I in Fig. 12-18).

The left-hand stub adds a susceptance in parallel with Y_3, and the amount of this susceptance must be enough to change the admittance Y_3 to Y_2. The points for Y_2 and Y_3 lie on the same constant-conductance circle of the chart (marked IV in the diagram) and hence the two admittances differ by susceptance only; the length of the stub line is adjusted to supply this difference of susceptance. Y_2 is the point at which Circle IV intersects Circle III. Circle III is similar to Circle I, but 180 degrees from it on the Smith chart for reasons that will shortly appear.

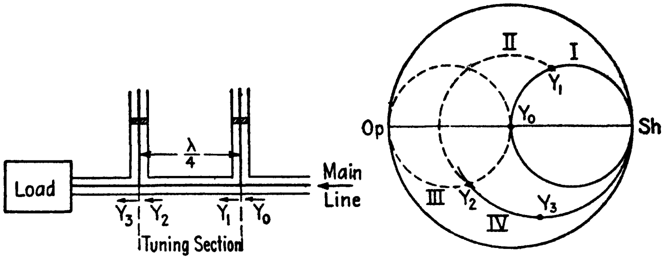

FIG. 12-18. Use of Smith chart for double-stub impedance matching.

The admittance Y_2 is the output admittance of the quarter-wavelength line between the two stubs. The input impedance Y_1 to this quarter-wavelength line is found by following around Circle II a distance corresponding to a quarter-wavelength of line (180 degrees on Circle II). Because of the construction specified for Circle III, Y_1 will necessarily fall on Circle I.

The right-hand stub adds susceptance in parallel with Y_1, By letting the input susceptance to this stub be equal and opposite to the susceptance component of Y_1, the admittance at the input to the tuning section is adjusted to a pure conductance equal to Y_0, thereby properly terminating the main line.

Practically, it is easier to adjust the length of two short-circuited stubs than the length and position of one. Although the double-stub tuner will not match any load to any line because of the restriction that Y_3 must lie outside of Circle I, difficulty can usually be avoided by moderate care in selecting the length of the line between the tuning section and the load.

A double-stub tuner with wider range of application has the two stub lines spaced $\frac{1}{8}$ wavelength apart. In general, closer spacing makes the excluded matching range of a double-stub tuner smaller, but it also makes the resonances in the tuner more extreme.

Triple-stub tuners can operate universally, to match any load to any line. There are only two adjustments, for the short-circuiting plungers in the first and third stubs are mechanically connected to move together. The stubs may advantageously be spaced a quarter-wavelength apart.

20. Other Impedance-matching Devices. A mere metallic pin or probe introduced into a coaxial line will alter the field configuration and make Z_0 discontinuous; it will therefore produce reflections and a pair of probes can be used for impedance matching. The penetration of the probes into the line must be adjustable.

The eccentric line has already been mentioned. This provides a method of adjusting the characteristic impedance of a quarter-wavelength line.

Another means of altering the characteristic impedance of a quarter-wavelength of line is to place a metal sleeve, $\frac{1}{4}$ wavelength long, on the inner conductor, thereby increasing its radius and changing the ratio of b/a. The same effect is obtained by using a metal sleeve that fits tightly within the outer conductor, reducing its inside diameter for a quarter-wavelength.

Two sleeves, one following the other, can be used as an adjustable device to produce a reflected wave equal and opposite to an undesired reflected wave from a mismatched load. Spacing between the sleeves is changed to give the correct magnitude of reflection, and the phase of the reflection is adjusted by moving both sleeves together along the line.

"Slugs" of dielectric material can be used instead of metal sleeves to change the characteristic impedance. The slugs fill the space between inner and outer conductors, and are effective because the characteristic impedance is inversely proportional to the square root of dielectric constant. The length of each slug is one-fourth of the wavelength of the wave in the dielectric material, which is less than that in air by the square root of the dielectric constant. Slugs produce a reflected wave, as do metal sleeves.

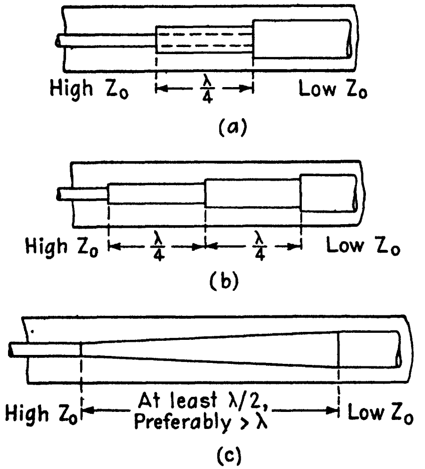

FIG. 12-19. Gradual transition of impedance.

All these methods of impedance matching have the common disadvantage of being frequency-sensitive. They operate correctly at one frequency only. The more different the impedances to be matched, the more frequency-sensitive the matching device may be expected to be. Consequently, an impedance transformation that is accomplished in a number of small steps will operate satisfactorily for a broader band of frequencies than will an impedance transformation accomplished in one large step.

Thus, a match between lines of different characteristic impedance can be accomplished at one frequency by using one quarter-wave sleeve as in Fig. 12-19*a*, but a transformation accomplished with two quarter-wave sleeves correctly proportioned (Fig. 12-19*b*) is less frequency-sensitive. This suggests the use of a tapered line, as in Fig. 12-19*c*.

C. TAPERED AND DISCONTINUOUS LINES

21. The Tapered Line. In considering a tapered line, it is necessary to retrace transmission-line analysis to the original formulation of the differential equations. In Chap. 1 it was assumed that z and y were not functions of x, that is, that they did not vary along the line. For the tapered line, z and y are required to be functions of x.

Hence, on differentiating the transmission-line equations,

$$\frac{dV}{dx} = zI \qquad \frac{dI}{dx} = yV \tag{12-102}$$

we obtain, for a tapered line,

$$\frac{d^2V}{dx^2} = z\frac{dI}{dx} + I\frac{dz}{dx} \tag{12-103}$$

To find dz/dx, the type of taper of the line must be defined.[23]

Let us select a so-called exponential taper:

$$z = z_1e^{-\delta x} \qquad y = y_1e^{\delta x} \tag{12-104}$$

The steepness of the taper is given by δ; z_1 and y_1 are the impedance and admittance per unit length of line at the end of the tapered section where $x = 0$. Note that the product zy is independent of x; this is practically true for any gradually tapered line, as it is for a uniform line, at the frequency of centimeter waves. Let us assume a lossless line, and write

$$zy = (j\omega l)(j\omega c) = -\omega^2lc = -\beta^2 \tag{12-105}$$

Differentiating Eqs. 12-104,

$$\frac{dz}{dx} = -\delta z \qquad \frac{dy}{dx} = \delta y \tag{12-106}$$

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[23] An approximate solution may be given for a line with arbitrary taper, if the taper is sufficiently gradual. See, for instance, J. C. Slater, "Microwave Transmission," Sec. 6 (Ref. 32).

Substitution of Eqs. 12-102 and 12-106 into 12-103 gives

$$\frac{d^2V}{dx^2} = zyV - \frac{1}{z}\frac{dV}{dx}\,\delta z \tag{12-107}$$

$$\frac{d^2V}{dx^2} + \delta\frac{dV}{dx} + \beta^2 V = 0 \tag{12-108}$$

A solution of this differential equation must be found. As a trial form, let us assume

$$V = V_1 e^{Sx} \tag{12-109}$$

in which V_1 and S are unknown. Substitution into Eq. 12-108 gives

$$S^2 + \delta S + \beta^2 = 0 \tag{12-110}$$

Equation 12-109 is therefore a correct solution, provided

$$S = -\frac{\delta}{2} \pm j\sqrt{\beta^2 - \left(\frac{\delta}{2}\right)^2} \tag{12-111}$$

(Only the incident wave, with the positive sign before the radical, will be retained in the discussion; but a reflected wave may also exist.) Voltage of the incident wave is

$$V = V_1 e^{-(\delta/2)x} e^{j\sqrt{\beta^2 - (\delta/2)^2}\,x} \tag{12-112}$$

V_1 is voltage at the end of the tapered section where $x = 0$.

Similarly, writing a differential equation in I, we obtain

$$I = I_1 e^{(\delta/2)x} e^{j\sqrt{\beta^2 - (\delta/2)^2}\,x} \tag{12-113}$$

These equations give voltage and current along the tapered line. It is seen that one is increasing and the other decreasing. The phase constant is of particular interest:

$$\text{Phase constant} = \sqrt{\beta^2 - \left(\frac{\delta}{2}\right)^2} \tag{12-114}$$

With a gradual taper, δ is small and the phase constant is nearly equal to β. With rapid taper, however, it is possible for $\delta/2$ to become greater than β; Eq. 12-114 then becomes imaginary. Waves cannot be propagated on a line with so sharp a taper, and if a section of line with such steep taper is used as a transition section for the purpose of matching impedance, it will behave as an abrupt discontinuity. Thus a tapered line acts as a high-pass filter, transmitting waves for which β is greater than $\delta/2$ and attenuating lower frequencies.

The characteristic impedance of a tapered line may be defined as V/I;

it is found by differentiating Eq. 12-109 and substituting the result in Eq. 12-102:

$$\frac{dV}{dx} = SV = zI \tag{12-115}$$

$$Z_0 = \frac{V}{I} = \frac{z}{S} = \frac{\beta}{\sqrt{\beta^2 - (\delta/2)^2} + j\delta/2}\sqrt{\frac{z}{y}} \tag{12-116}$$

This, the characteristic impedance of the tapered line, reduces to the usual value for Z_0 if there is no taper, δ being zero. For a tapered line, however, Z_0 is a function of distance along the line. Also, β being proportional to frequency, Z_0 is a function of frequency. Moreover, Z_0 for a tapered line is not real, but complex, even for a transmitted frequency.

If the characteristic impedance at the end of the tapered line, where $x = 0$, is Z_{01}, we may combine Eqs. 12-116 and 12-104 to give

$$\frac{Z_0}{Z_{01}} = e^{-\delta x} \tag{12-117}$$

If x is the total length of the taper, this gives the ratio of output to input impedance, and hence the impedance transformation of the tapered section.

Conclusions. From the above development, it will be seen that a tapered line provides an impedance transformation that is nearly independent of frequency for frequencies well above the cutoff value at which $\beta = \delta/2$. It is well to have the lowest frequency that must be transmitted ten times the cutoff frequency. The tapered section then produces little reflection of any transmitted frequency.

Practically, this means that a 2 to 1 change of characteristic impedance may be accomplished by using a tapered section that is 1 wavelength long. A good impedance match results at this frequency and at all higher frequencies.

Results are fairly good if the tapered section is as little as $\frac{1}{2}$ wavelength long, particularly if the impedance ratio required is less than 2 to 1. It is not worth while to use a tapered section much shorter than $\frac{1}{2}$ wavelength, for a taper that is $\frac{1}{4}$ wavelength or shorter will give nearly as much reflection as will the abrupt change of impedance at a mismatch.

Those frequencies for which the length of the taper is a multiple of a half-wavelength are transmitted with the least reflection. This may be a guide in choosing the length of taper.

Finally, it is found by measurement as well as by approximate computation that a simple conical taper is nearly as good as the exponential taper. The above analysis applies almost equally well, provided the taper is as gradual as it must be in practice. Practically, therefore, a tapered line is made to have the spacing between conductors vary linearly with distance, and the length of the taper is about one-half to one wavelength of the lowest frequency.

22. Susceptance at Discontinuities. A short taper between two sections of line, short compared to a wavelength, must be distinguished from the relatively long taper discussed in the previous section. A short taper is merely a convenient form of discontinuity. It is convenient because it can be designed to introduce little if any excess susceptance into the line.

A discontinuity in a line introduces distortion of the electric and magnetic fields that may be interpreted as the addition of a lumped susceptance shunted across the line. If one coaxial line feeds into another coaxial line of larger size, as in Fig. 12-20*a*, even though diameters are adjusted to give both lines the same characteristic impedance, there is a combination of electric and magnetic fields at the junction that is different from the pattern elsewhere along the line; the relation between the energy stored in the magnetic field and the energy stored in the electric field is different in the neighborhood of the junction. If there is an excess of magnetic energy storage, the distortion of the field at the junction is equivalent to shunt inductive susceptance; if there is an excess of electric energy storage, to shunt capacitive susceptance.

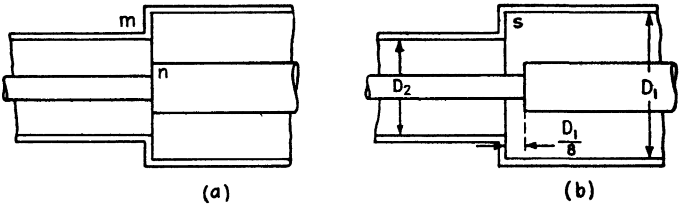
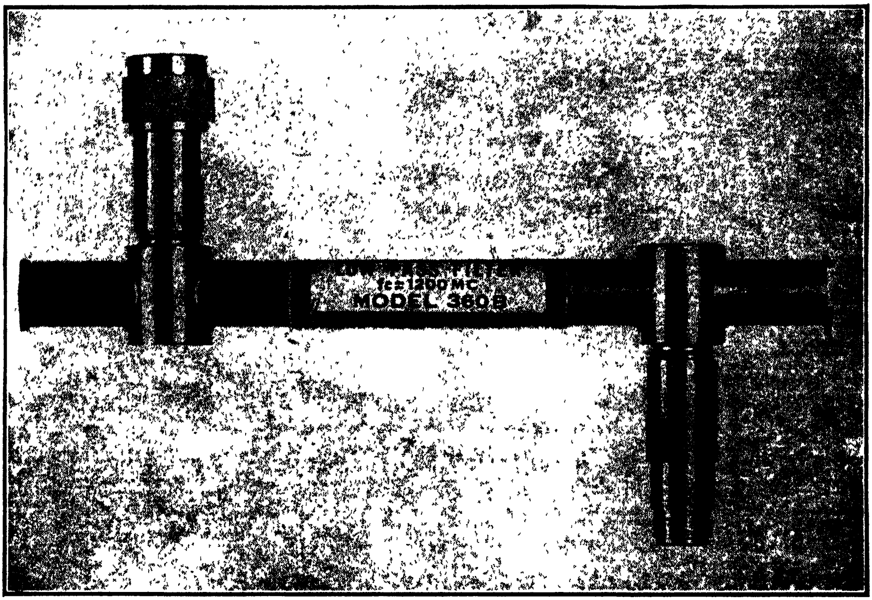


FIG. 12-20. Spacing to minimize shunt susceptance.

The susceptance introduced at such a junction is usually negligible except at ultra-high frequencies; even for high radio frequencies the effect can be disregarded. But for centimeter waves, irregularities that distort the field pattern must be given consideration. The ends of the line are discontinuities that must always exist, and many lines have bends and corners. Probes, supports, or other objects projecting into the fields of the line also introduce excess susceptance, usually capacitive (see Secs. 13 and 14, Chap. 13).

To avoid susceptance at a discontinuity, it is necessary to keep the stored energy of electric and magnetic fields in the same proportion in the vicinity of the discontinuity as elsewhere along the line. Thus, if a coaxial line is joined to another line of larger size, as in Fig. 12-20*a*, there is distortion of the electromagnetic field around the edges *m* and *n*, causing partial reflection of an incident wave at the junction even though both lines have the same Z_0. If, however, the junction of the inner conductor is displaced slightly from the junction of the outer conductor, as in Fig. 12-20*b*, such reflection can be minimized. The figure gives approximately the best relation of dimensions.[24] Alternatively, a short taper could be used to connect one line to the other, the length of the taper preferably not exceeding $\frac{1}{16}$ wavelength.

Shunt susceptance in the junction of Fig. 12-20*a* results from strengthening the electric field between the corners *m* and *n*. In *b*, these corners are farther apart, and magnetic field in the additional space *s* increases the inductive effect, thus maintaining the proper balance of electric and magnetic energy storage.



A decimeter-wave filter (cutoff frequency 1,200 megacycles) for coaxial-line attachment. The filter employs transmission-line sections, including a stub at each end, and discontinuity capacitances, as reactive elements. (*Courtesy of Hewlett-Packard Company.*)

If shunt susceptance cannot be avoided at a discontinuity, its effects can be eliminated by introducing an equal susceptance at a distance of $\frac{1}{4}$ wavelength. A tuning stub, a screw projecting into the line, or other convenient means of introducing susceptance can be used. Both susceptances then produce reflections; the reflected waves, being equal in magnitude and opposite in phase, add to zero. Adjustment is best done experimentally, tuning to eliminate standing waves.

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[24] From "Non-reflecting Step Sections between Lines of Equal Characteristic Impedance," by H. H. Meinke. This relation is said to be satisfactory in practical application for "all usual frequencies," for lines with Z_0 below 80 ohms, and for ratios of D_1/D_2 less than 4.

The refinement of tuning out the susceptance of junctions between two coaxial lines of equal Z_0 is not necessary if the diameters of the lines are much smaller than the transmitted wavelength. With hollow wave guides, however, the diameter of the guide is necessarily of the same order of magnitude as the wavelength, and the susceptance of discontinuities is always a problem. This subject will be discussed in the next chapter.

PROBLEMS

12-1. A coaxial line consists of a copper tube 20 millimeters in diameter and an inner conductor one-fourth as large. What (approximately) is the highest frequency for which this line can well be used?

12-2. How large a coaxial line is suitable for transmitting signals at a 3,000-megacycle frequency?

12-3. Derive the expressions of Fig. 12-1 for Z_0 of a four-wire line, and for Z_0 of a coaxial line including the effect of dielectric spacers.

12-4. Derive Z_0 of a double coaxial line (see Fig. 12-1) including the effect of dielectric spacers.

12-5. What is the "effective depth of penetration" in copper of current at 4,000-megacycle frequency? At this frequency, what is the resistance per thousand feet of No. 18 AWG copper wire? What is the ratio of this high-frequency resistance to the d-c resistance of the wire?

12-6. What is the "effective depth of penetration" in silver of current at 3,000-megacycle frequency? Compare this (if possible) with the thickness of ordinary silver plating.

12-7. Find decibels attenuation in 10 meters of copper coaxial line at 4,500 megacycles. The tube inside diameter is 8.0 millimeters, the wire diameter 2.2 millimeters, and the dielectric is air. Find the attenuation in a line of similar dimensions with polystyrene dielectric (relative dielectric constant 2.55; power factor 0.0006).

12-8. Find decibels attenuation in 50 meters of copper coaxial line at 1,000 megacycles. The tube inside diameter is 1.00 centimeter and the wire diameter is 0.278 centimeter; the dielectric material is air. Find the attenuation in a line of similar dimensions with rubber insulation (relative dielectric constant 2.80; power factor 0.005).

12-9. It is stated that attenuation in a coaxial line is within 10 per cent of minimum if b/a is between 2.3 and 6. Find Z_0 for $b/a = 2.3$ and $b/a = 6$.

12-10. Plot attenuation in an air-dielectric coaxial line as a function of b/a (with b constant), and note the value for minimum attenuation.

12-11. Plot input impedance to a quarter-wavelength short-circuited coaxial line with air dielectric as a function of b/a (with b constant). What value gives maximum impedance, and what is the value of Z_0 that corresponds to maximum impedance?

12-12. Show that $Q = \omega l/r$ for a line with no dielectric loss.

12-13. It is stated that $Q = 1{,}000$ is reasonable for a solid-dielectric line of ordinary size at frequencies near 3,000 megacycles. Assuming a dielectric material of good quality (see table, page 309), is this statement correct?

12-14. It is stated that $Q = 10{,}000$ can be attained at microwave frequencies for a coaxial air-dielectric line. Is this possible with a wavelength of 10 centimeters and a line small enough to prevent the propagation of detrimental modes?

12-15. An air-filled coaxial line is made of a 10.0-millimeter inside-diameter tube and a 3.6-millimeter-diameter wire, both copper. The receiving end of the line is short-circuited and, operating at 1,000 megacycles, the line is adjusted to quarter-wavelength resonance. Find the theoretical input impedance. Give both Z and Z/Z_0.

12-16. Repeat Prob. 12-15 for a line with the same size outer conductor and an inner conductor of 1.0-millimeter diameter.

12-17. Find the admittance to the line of Prob. 12-15 if it is $\frac{1}{2}$ wavelength long and short-circuited at the far end (both Y and Y/Y_0).

12-18. Find the admittance to the line of Prob. 12-16 if it is $\frac{1}{2}$ wavelength long and short-circuited at the far end (both Y and Y/Y_0).

12-19. Tabulate the impedance properties (as infinite, high, medium, low, zero; resistance, inductance, capacitance) of quarter-wavelength, half-wavelength, and three-quarter-wavelength lines, both open and short-circuited at the far end, for fundamental frequency, second and third harmonics, subharmonics of one-half and one-third fundamental frequency, and for direct current.

12-20. It is stated that "Eqs. 12-43 and 12-44 apply also to an open-circuited line that is $\frac{1}{2}$ wavelength long." Verify this statement.

12-21. What is the optimum value of b/a to give high admittance to an open-circuited quarter-wavelength line? to a short-circuited half-wavelength line?

12-22. Design a supporting stub for a 2,800-megacycle radar transmission line. The line is proportioned for maximum power transmission. (Do not attempt *broadbanding*.)

12-23. Repeat Prob. 12-22 for a 1,500-megacycle radar transmission line.

12-24. A wave in a coaxial line (frequency, 1,000 megacycles) passes from a region of air dielectric to a region of polyethylene dielectric (as when a dielectric *slug* is encountered). Find the reflection coefficient at the polyethylene face.

12-25. Repeat Prob. 12-24 for a wave passing from polystyrene into air (as when leaving a dielectric *slug*).

12-26. Considering the space between plunger and tube, from b to c in Fig. 12-6, to be a coaxial transmission line of small clearance, find its characteristic impedance. Diameter of the tube is 0.75 inch; clearance to plunger is 5 mils (0.005 inch).

12-27. Repeat Prob. 12-26 to find Z_0 for the "transmission line" between the plunger and the inner conductor. Diameter of conductor is 0.25 inch; clearance to plunger is 5 mils.

12-28. Derive Eqs. 12-50, 12-52, and 12-53 for impedance of an open-circuited line.

12-29. Estimate a rough value of radiation resistance for a line composed of two copper conductors, each 0.50-inch diameter and 2.50-inch spacing (center to center), 12 feet long, at 300 megacycles frequency. Compare with the characteristic impedance of the line. Why is the comparison with characteristic impedance interesting?

12-30. Find the standing-wave ratio resulting from partial reflection at the polyethylene face in Prob. 12-24. Find both σ (VSWR) and η (PSWR).

12-31. Find the standing-wave ratio resulting from partial reflection at the polystyrene-air surface in Prob. 12-25.

12-32. If Probs. 2-9 and 2-10 have not previously been solved, they may now be done. Use the method of Sec. 5, Chap. 2.

12-33. If $Z_r = 2Z_0$, $\alpha x = 0.33$, and $\beta x = 2.15$, compute input impedance by the method of Sec. 5, Chap. 2.

12-34. An antenna (Fig. 12-16) produces on a transmission line a standing-wave ratio of 2.0, with a voltage minimum 0.08 wavelength from the antenna. By computation (without the use of charts), find (*a*) the antenna impedance and (*b*) the reflection factor at the antenna terminals.

12-35. A lossless line is terminated in a load such that $Z_r/Z_0 = 0.98 + j0.70$. The line is 0.55 wavelength long. Using the method of Sec. 5, Chap. 2, compute the input impedance.

12-36. A lossless line that is $\frac{5}{16}$ wavelength long is terminated in a load such that $Z_r/Z_0 = 0.6 + j0.8$. Find the input impedance, computing by the method of Sec. 5, Chap. 2.

12-37. Repeat Prob. 2-9 using the Smith chart. (For this and the following problems, printed copies of the Smith chart should be obtained.)

12-38. Repeat Prob. 12-36 using the Smith chart.

12-39. What is the input impedance to a line 0.35 wavelength long if the load is purely resistive and the standing-wave ratio is 2.5? The load is at a voltage maximum.

12-40. What is the input impedance to a line 0.70 wavelength long if the load is purely resistive and the standing-wave ratio is 1.75? The load is at a voltage minimum.

12-41. The standing-wave ratio on a line is 1.25, and there is a voltage minimum 0.75 wavelength from the load. What is the load impedance?

12-42. The standing-wave ratio on a line is 1.68, and there is a voltage minimum 0.38 wavelength from the load. What is the load impedance?

12-43. A line, being terminated in its characteristic admittance, is *flat* for a distance of 0.25 wavelength from the load to a junction point at which an additional admittance, equal to $0.75Y_0$, is shunted across the line. The line extends for 0.375 wavelength beyond the junction point to its input terminals. What is the input admittance?

12-44. Repeat Prob. 12-43, taking the load at the extreme end of the line to be $2Y_0$ instead of Y_0.

12-45. In Prob. 12-44, what admittance at the junction point would make the standing-wave ratio from the junction point to the input equal 1.00?

12-46. In Prob. 12-44, where could a purely reactive admittance be connected, and what would be its value, to make the standing-wave ratio from the point of connection to the input equal 1.00?

12-47. A coaxial line has solid dielectric material with a relative dielectric constant of 2.1 and a power factor of 0.0022. The line is 150 centimeters long; frequency is 4,550 megacycles. $Z_0 = 50$ ohms and $Z_r = 150$ ohms (resistance). What is the input impedance to the line?

12-48. Repeat Prob. 12-47 with $Z_r = 40$ ohms (resistance).

12-49. A load has admittance $Y_r = (1.50 + j0.25)Y_0$. Find the length and location for a single short-circuited tuning stub.

12-50. A load has admittance $Y_r = (0.75 + j0.48)Y_0$. Find the length and location for a single short-circuited tuning stub.

12-51. Design a double-stub tuner for the line of Prob. 12-49. Make the distance between stubs $\frac{1}{4}$ wavelength, and specify distance from load to tuner as well as the length of each stub.

12-52. Design a double-stub tuner for the line of Prob. 12-50. Make the distance between stubs $\frac{1}{4}$ wavelength, and specify distance from load to tuner as well as the length of each stub.

12-53. For what range of admittances (Y_3 in Fig. 12-18) can the double-stub tuner with quarter-wavelength distance between stubs not be used? If the distance from the load to the tuning section can be determined as part of the design, to what extent is this limitation removed?

12-54. Explain the statement: "In general, closer spacing makes the excluded matching range of a double-stub tuner smaller, but it also makes the resonances in the tuner more extreme." For what range of admittances (at Y_3) can the double-stub tuner with eighth-wavelength spacing not be used?

12-55. A tuning slug of polyethylene is placed in a coaxial line that would otherwise be flat. The slug, which completely fills the space between conductors, is 1.00 centimeter long; the wavelength in air is 10.0 centimeters. What is the standing-wave ratio after insertion of the slug?

12-56. A tuning sleeve of copper is placed in a coaxial line that would otherwise be flat. For the line, $b = 4.0$ millimeters, $a = 1.0$ millimeter. The sleeve fits tightly about the inner conductor, and its outer radius is 2.0 millimeters. If the sleeve is 1.50 centimeters long and the wavelength is 10.0 centimeters, what is the standing-wave ratio after insertion of the sleeve?

12-57. Show that an exponentially tapered line that gives a 2 to 1 change of characteristic impedance in one wavelength satisfies the condition that "the transmitted frequency should be more than ten times the cutoff frequency."

12-58. If the frequency applied to an exponentially tapered line is 10 times the cutoff frequency, how does the wavelength λ on the tapered line compare with the wavelength λ_0 of the same frequency on a uniform line?

12-59. A uniform coaxial line leads into a tapered line (assume exponential taper). At the point of junction, the conductors of the tapered line are the same size as the conductors of the uniform line. Frequency is ten times the cutoff frequency of the tapered line. Find the reflection factor at the junction.

12-60. What is the standing-wave ratio on the uniform line of Prob. 12-59 if the only reflected wave is that from the junction in question? At what distance from the junction is there a voltage maximum (what fraction of a wavelength)?

CHAPTER 13

INTRODUCTION TO WAVE GUIDES

1. Hollow Wave Guides. All transmission lines are truly wave guides, for their purpose is to guide the energy of electromagnetic waves from place to place, but in common usage the term *wave guide* is applied to a hollow tube. Such a tube may be of circular cross section, but is more commonly rectangular, and electromagnetic waves that enter one end of the tube are transmitted to the other where their energy is utilized.

It is easy to see the relation of a wave guide to a coaxial line. Figure 13-1*a* shows the electric field that exists at a particular instant in a coaxial line; a traveling wave is passing from left to right, and a little more than one full wavelength is shown. Lines of the electric field are radial and terminate on the inner and outer conductors. Magnetic flux lines are circles surrounding the inner conductor; the magnetic lines are not shown in the diagram, but points at which they penetrate a vertical plane through the axis are indicated.

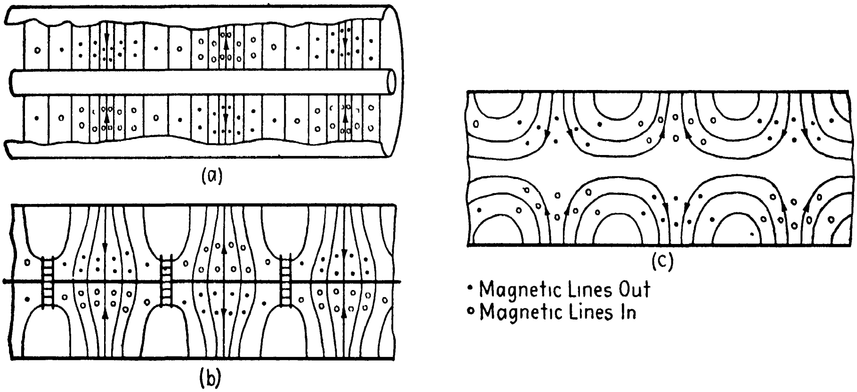

FIG. 13-1. Electric field in a coaxial line and in a hollow wave guide.

In Fig. 13-1*b*, it is assumed that a large number of small condensers have been placed in the inner conductor of the coaxial line. This may be thought of as a capacitance-loaded coaxial line. Electric field between the inner and outer conductors is still primarily radial but now there is also, between the plates of the condensers, an axial field. Much as in the coaxial line of Fig. 13-1*a*, current flows in the sections of inner conductor that connect condenser to condenser, but there is no conduction current through the space between the plates of the condensers. Through the space between the plates of the condensers there is *displacement current*. Displacement current produces magnetic field just as does conduction current,[1] and the magnetic field of Fig. 13-1*b* is much like the magnetic field of Fig. 13-1*a*.

If the number of condensers in the inner conductor is now considered to be increased, and still further increased, the limiting condition will be a series of condensers with no inner conductor remaining. This does not differ in any essential way from a hollow tube, as in Fig. 13-1*c*. Such a hollow tube has no conduction current along the axis, but the axial component of electric field (through what may be looked upon as the distributed capacitance down the center of the tube) provides displacement current that is entirely equivalent to conduction current. Thus the electric and magnetic field patterns in the hollow guide of Fig. 13-1*c* are not essentially different from those in the coaxial line of Fig. 13-1*b*.

2. High-pass Filter Characteristics of Wave Guide. We have seen in previous chapters that a transmission line loaded with lumps of inductance will not pass high frequencies. Similarly, a line loaded with capacitance, or a hollow wave guide, will not pass low frequencies.

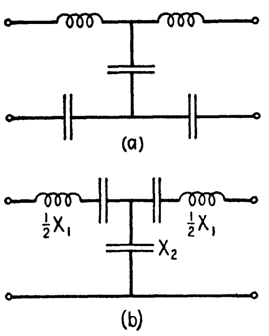

FIG. 13-2. T networks equivalent to a short section of circular wave guide (TM$_{0,1}$ mode).

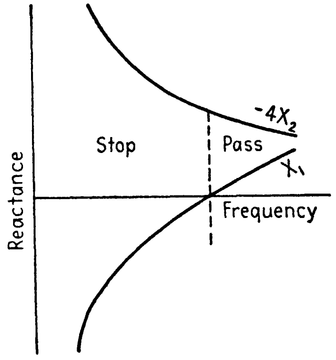

FIG. 13-3. Showing cutoff frequency of a wave guide.

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[1] This was first stipulated by Maxwell and constitutes one of *Maxwell's equations*. *Displacement current* is not an actual current but is a quantity proportional to the rate of change (with time) of the electric field. See "Fundamentals of Electric Waves," Chap. VIII (Ref. 29). Those interested in the history, but not the mathematics, may also see "Exploring Electricity" (Ref. 28).

A T network equivalent to a short length of the capacitance-loaded wave guide of Fig. 13-1*b* is shown in Fig. 13-2*a* or *b*. Remembering the filter criterion of Chap. 10: "In the pass band, X_1 and X_2 must be of opposite sign, and $4X_2$ must be numerically greater than X_1," we plot X_1 and $-4X_2$ of Fig. 13-2 in Fig. 13-3. The curve for X_1 has the familiar form of reactance of a series resonant circuit, and that for $-4X_2$ is hyperbolic. The low-frequency band is a stop band; the signs are such that there must be attenuation, not transmission. Above the cutoff frequency the network transmits.[2]

3. The Parallel-strip Transmission Line. Although circular wave guides are sometimes useful, particularly when ultra-high-frequency energy is to be carried through a rotating joint (as, for example, to a rotating antenna), wave guides of rectangular cross section are more common. A rectangular wave guide is related to a parallel-strip transmission line.

Consider a transmission line of two conductors, the conductors being parallel strips or ribbons of metal instead of wires. Figure 13-4*a* shows a pair of such strips, and indicates the electric and magnetic fields between them. The magnetic field, of course, does not cease at the edge of the strips, but the flux lines lying beyond and above the strips are omitted to avoid confusion. Arrows indicate current in the upper strip; current in the lower strip, directly below, is equal and opposite.

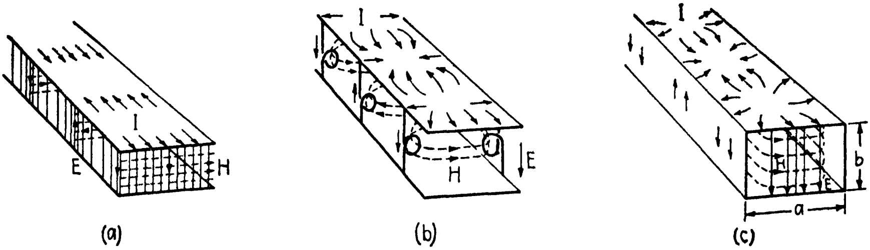

FIG. 13-4. Current distribution and fields in a parallel-strip line and in a rectangular wave guide.

Now consider that a number of inductive connections are made between the upper and lower strips, as in *b* of the same figure. These connections are suggested in the drawing as pieces of wire with a twist to indicate the inductance, although as a matter of fact straight pieces of wire would similarly have inductance.

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[2] At a sufficiently high frequency, out of the range of Fig. 13-3, X_1 becomes numerically greater than $4X_2$, indicating that the lumped network of Fig. 13-2 has another stop band. This has no meaning for a wave guide, however, for the T network of Fig. 13-2 is truly equivalent to the wave guide only in the limit as the size of each lump becomes vanishingly small and the number of T sections becomes infinite. Figure 13-3 is modified as the limit is approached in such a way that, although f_c is not changed, X_1 becomes lower and X_2 greater at high frequencies, and in the limit the curves do not cross. Thus the wave guide is a high-pass, not a band-pass, filter.

These inductive connections between the top and bottom conductors of the transmission line will carry current. Except for their inductance they would short-circuit the line, but at sufficiently high frequency the inductance prevents a short-circuiting effect. Current in these inductive connections produces a component of magnetic field parallel to the axis of the transmission line—an axial component. This axial component of field, combined with the transverse component produced by current in the upper and lower strips, causes magnetic flux to appear in loops between the conducting strips, as indicated in the diagram. The magnetic flux pattern is simplified in this diagram, and parts of the magnetic field that are not essential in the present discussion are omitted.

Lines of electric field are not shown in Fig. 13-4*b*, as they are substantially the same as in Fig. 13-4*a* and would tend to confuse the diagram. Lines of current flow are shown in the upper strip of the parallel-strip line; note that some of the current that formerly flowed back and forth in the upper strip is now carried by the inductive connections between the upper strip and the lower strip.

Finally, consider that the number of inductive connections along the edges of the parallel-strip line is increased (straight connecting wires are now assumed, the kink being unnecessary) until, in the limit, the rows of connecting wires become indistinguishable from vertical walls of metal; the result is a rectangular wave guide as in Fig. 13-4*c*. The wave guide has no external field, for the magnetic flux lines are closed loops within the guide as a result of the axial magnetic field produced by current in the vertical side walls of the guide.

4. Differential Equations for the Rectangular Guide. As Fig. 13-2 shows an equivalent T network for the circular wave guide, so Fig. 13-5 shows an equivalent network for a short section of the rectangular guide. (Both equivalent circuits apply for only the specified patterns of electromagnetic field in the guides.) Inductance of the side walls is represented by l', which shunts the capacitance of the line from one conductor to the other, as is most clearly seen in Fig. 13-4*b*. It could be shown, using the method of Fig. 13-3, that this network attenuates low frequencies and transmits high frequencies—the rectangular wave guide, like the circular wave guide, is a high-pass filter—but more use-

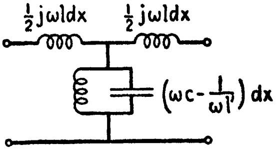
$\frac{1}{2}j\omega l dx$
$\frac{1}{2}j\omega l dx$
$\left(\omega c - \frac{1}{\omega l'}\right) dx$

FIG. 13-5. T network equivalent to a short section of rectangular wave guide ($TE_{1,0}$ mode).

ful quantitative results can be obtained by computing operation from the differential equations.

Differential equations as for a transmission line are used, with a term added to represent the shunt inductance of the side walls. Starting with the transmission-line equations 1-6 and 1-16, let us (for the present) neglect all losses and set r and g equal to zero. Then, introducing l' as a shunt inductance in parallel with the shunt capacitance, as in Fig. 13-5, we have

$$\frac{dV}{dx} = j\omega lI \tag{13-1}$$

$$\frac{dI}{dx} = j\left(\omega c - \frac{1}{\omega l'}\right)V \tag{13-2}$$

The first step in solving these equations is to differentiate again, obtaining

$$\frac{d^2V}{dx^2} = j\omega l\frac{dI}{dx} = -\omega l\left(\omega c - \frac{1}{\omega l'}\right)V = -\left(\omega^2 lc - \frac{l}{l'}\right)V \tag{13-3}$$

It will be recalled (and may easily be proved by substitution) that a solution of this differential equation is

$$V = V_1 e^{\gamma x} \tag{13-4}$$

where

$$\gamma = \sqrt{-\left(\omega^2 lc - \frac{l}{l'}\right)} = j\sqrt{\omega^2 lc - \frac{l}{l'}} \tag{13-5}$$

Under conditions that make γ purely imaginary, this is recognized as describing a traveling wave.

If, as before, we let $\gamma = \alpha + j\beta$, the phase constant β may be written

$$\beta = \sqrt{\omega^2 lc - \frac{l}{l'}} \tag{13-6}$$

This is for the incident traveling wave; there is also a solution using the negative sign for the radical of γ and β that accounts for the reflected wave. It is sufficient to consider only the incident wave at present.

To have β real, permitting wave propagation, it is necessary that

$$\omega^2 lc - \frac{l}{l'} > 0 \tag{13-7}$$

or

$$\omega^2 l'c > 1 \tag{13-8}$$

This condition is satisfied by a sufficiently high value of frequency. It is also satisfied at any given frequency if the product $l'c$ is large enough.

Referring to Fig. 13-4, the capacitance c between the upper and lower strips can be increased by making the height b less, or the width a greater. Since l' represents the inductance encountered by current flowing vertically in the side walls, the value of l' is increased by increasing the height b and also by increasing the width a; the latter because it allows more space within the guide for the axial magnetic field. It works out that an increase of height b has no net effect on the product $l'c$, but an increase of width a increases both l' and c. Hence the condition of Eq. 13-8 for wave propagation can be satisfied at any frequency by making the width a of the wave guide sufficiently great.

Thus, a guide of any width has a cutoff frequency below which it will not transmit waves. The cutoff frequency is not affected by the height b of the guide, but is inversely proportional to the width a.

5. Cutoff Frequency. The cutoff frequency is the lowest frequency for which β is real. By Eq. 13-6,

$$\beta = \sqrt{\omega^2 lc - \frac{l}{l'}} \tag{13-9}$$

and the cutoff frequency f_c is the frequency that makes this expression zero:

$$f_c = \frac{1}{2\pi\sqrt{l'c}} \tag{13-10}$$

To compute the cutoff frequency, l' and c are needed. These are equivalent inductance and capacitance, introduced in the equivalent circuit of Fig. 13-5. They cannot be evaluated from the dimensions of the guide without knowing the distribution of electric and magnetic fields within the guide. This distribution of fields is obtained from electromagnetic field theory and we can best proceed by accepting, from field theory, without attempting a derivation in this book,[3] that

$$\beta = \sqrt{\omega^2 \mu\epsilon - \frac{\pi^2}{a^2}} \tag{13-11}$$

Since (as in Eq. 6-75) $\mu\epsilon = lc$, comparison of Eq. 13-11 with 13-9 gives

$$\frac{l}{l'} = \frac{\pi^2}{a^2} \tag{13-12}$$

We can now compute cutoff frequency. Equation 13-12 is used to eliminate l' from Eq. 13-10, giving

$$f_c = \frac{1}{2a\sqrt{lc}} \tag{13-13}$$

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[3] See, for instance, "Fundamentals of Electric Waves," Chap. XIII (Ref. 29).

Remembering that l and c refer to the transmission line of Fig. 13-4a on which waves are transmitted with the velocity $1/\sqrt{lc}$, we may write

$$f_c = \frac{v_0}{2a} \tag{13-14}$$

where v_0 is the velocity of transmission of a wave on a transmission line.

If λ_c is the wavelength that a wave of frequency f_c would have on a transmission line, or traveling as a simple plane wave, $\lambda_c = v_0/f_c$ and Eq. 13-14 becomes

$$\lambda_c = 2a \tag{13-15}$$

This may be stated in words: *For a signal to be transmitted through a hollow rectangular guide, the width of the guide must be greater than half the free-space wavelength of the signal.*

6. The Guide Factor. For a lossless transmission line, β is $\omega\sqrt{lc}$. If this transmission-line value of β is designated β_0, we write

$$\beta_0 = \omega\sqrt{lc} \tag{13-16}$$

For a rectangular waveguide, β is (by Eq. 13-9)

$$\beta = \omega\sqrt{lc}\sqrt{1 - \frac{1}{\omega^2 l'c}} \tag{13-17}$$

Comparing these two, it is convenient to write

$$\beta = \beta_0 G \tag{13-18}$$

with

$$G = \sqrt{1 - \frac{1}{\omega^2 l'c}} \tag{13-19}$$

This constant G appears repeatedly in the discussion of wave-guide operation, and it will be helpful to examine it before going further. It may be written in various forms. Using Eq. 13-12 to eliminate l' from G, and using $v_0 = 1/\sqrt{lc}$, as in Eq. 13-14,

$$G = \sqrt{1 - \frac{v_0^2}{4f^2a^2}} \tag{13-20}$$

An alternative form is given by Eq. 13-14:

$$G = \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{13-21}$$

Equation 13-21, here derived for a rectangular wave guide, is generally applicable to guides of any form and to any mode of wave transmission.

Numerical values for G are plotted in Fig. 13-6. To use this diagram, values of f_c/f must be known. For a rectangular guide, using Eq. 13-14,

$$\frac{f_c}{f} = \frac{v_0}{2af} \qquad (13\text{-}22)$$

If λ_0 designates the wavelength that a signal of frequency f would have on a transmission line, $\lambda_0 = v_0/f$, and for a rectangular guide of width a (the electromagnetic field pattern or mode being as in Fig. 13-4c);

$$\frac{f_c}{f} = \frac{\lambda_0}{2a} \qquad (13\text{-}23)$$

This gives an easy means of computing f_c/f, and hence of computing G.

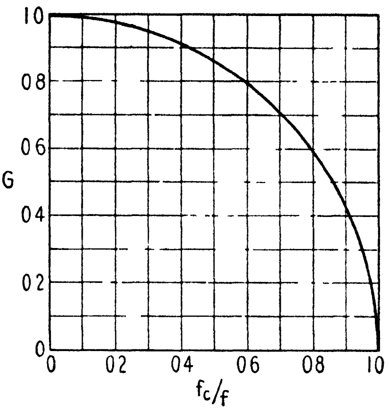

FIG. 13-6. The guide factor G. For values of f_c/f, see Table 13-2.

7. Phase Velocity and Wavelength. The velocity of a wave in a hollow guide may be quite different from the velocity of a wave on a transmission line.[4] Since the velocity of a wave is ω/β, we have, from Eq. 13-18,

$$v_\varphi = \frac{\omega}{\beta_0 G} = \frac{v_0}{G} \qquad (13\text{-}24)$$

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[4] Assuming air- or gas-filled wave guides, *free-space wavelength* is equivalent to *wavelength on a transmission line*. Strictly, v_0 is velocity of wave propagation on a lossless transmission line that has the same dielectric material as the wave guide under consideration. If this dielectric is air or other gas, as is usual in wave guides, v_0 is the velocity of propagation of plane waves in air, or practically in free space, and is 3×10^8 meters per second. Symbols used in this discussion may be tabulated:

f is operating frequency
f_c is cutoff frequency
v_0 is velocity of wave propagation on a lossless transmission line
v_φ is phase velocity in the wave guide
v_g is group velocity in the wave guide
λ is the wavelength of frequency f in the guide
λ_0 is the wavelength of frequency f on a transmission line (or in free space), equal to v_0/f
λ_c is the wavelength on a transmission line (or in free space) of the cutoff frequency f_c; it is v_0/f_c
β is phase constant at frequency f in the wave guide
β_0 is phase constant at frequency f on a transmission line

Velocity is written v_φ to indicate that it is *phase* velocity: phase velocity is the velocity at which waves would appear to travel along a guide if the electric field were visible. (The velocity that has been considered for waves on transmission lines in previous chapters has in all cases been phase velocity, but until now there has been no necessity to distinguish it by a special subscript.)

Phase velocity in a guide is always greater than transmission-line velocity v_0, for G is always less than one. At high frequencies, far above cutoff, v_φ is nearly equal to the transmission-line velocity, but if the operating frequency approaches the cutoff frequency the phase velocity increases without limit.

The wavelength of a signal within the guide is found by dividing velocity by frequency:

$$\lambda = \frac{v_\varphi}{f} = \frac{v_0}{fG} = \frac{\lambda_0}{G} \tag{13-25}$$

Thus wavelength in the guide is greater than free-space wavelength in inverse proportion to G; the difference is negligible at frequencies far beyond cutoff, for which G is nearly 1, but near cutoff frequency the wavelength in the guide is quite great.

At cutoff frequency (and below), the wavelength is infinite. This means that the signal has the same phase at all points along the guide; it does not travel as a wave, but merely produces a pulsating field that cannot travel (see Sec. 9).

We have seen that several of the more important parameters of wave-

TABLE 13-1

WAVE-GUIDE FORMULAS

Air-filled Coaxial Line	*Air-filled Wave Guide*
$v_0 = 3 \times 10^8$ meters per second	$v_\varphi = \dfrac{v_0}{G}$
$\lambda_0 = \dfrac{3 \times 10^8}{f}$ meters	$\lambda = \dfrac{\lambda_0}{G}$
$\beta_0 = \dfrac{\omega}{v_0} = \dfrac{2\pi f}{3 \times 10^8}$	$\beta = \beta_0 G$
	$v_g = v_0 G$

f is frequency in cycles per second.
f_c is cutoff frequency (see Table 13-2).
$G = \sqrt{1 - (f_c/f)^2}$ (see Fig. 13-6).

guide operation differ from those of a coaxial line by the factor G. Equations for these are collected in Table 13-1. Also, various convenient formulas are given in Table 13-2.

8. Group Velocity. One of the formulas given in Table 13-1 is for v_g, the group velocity of a signal in a guide. Phase velocity is the velocity

TABLE 13-2

PARAMETERS OF WAVE GUIDES

(Air-filled)

	Cutoff frequency, f_c	Cutoff free-space wavelength, λ_c	$\frac{f_c}{f}$	Guide factor G (see Fig. 13-6)
Rectangular Guide:				
Dominant mode ($TE_{1,0}$)	$\frac{1.5 \times 10^8}{a}$	$2a$	$\frac{\lambda_0}{2a}$	$\sqrt{1 - \frac{\lambda_0}{2a}}$
Next higher modes if $a = 2b$ ($TE_{0,1}$ and $TE_{2,0}$)	$\frac{3 \times 10^8}{a}$	a	$\frac{\lambda_0}{a}$	$\sqrt{1 - \frac{\lambda_0}{a}}$
Circular Guide:				
Dominant mode ($TE_{1,1}$)	$\frac{1.76 \times 10^8}{d}$	$1.71d$	$0.586 \frac{\lambda_0}{d}$	$\sqrt{1 - 0.586 \frac{\lambda_0}{d}}$
Symmetrical mode ($TM_{0,1}$)	$\frac{2.30 \times 10^8}{d}$	$1.305d$	$0.77 \frac{\lambda_0}{d}$	$\sqrt{1 - 0.77 \frac{\lambda_0}{d}}$
Next higher mode ($TE_{2,1}$)	$\frac{2.92 \times 10^8}{d}$	$1.030d$	$0.97 \frac{\lambda_0}{d}$	$\sqrt{1 - 0.97 \frac{\lambda_0}{d}}$
Coaxial Line:				
Usual mode (TEM)...	0	∞	0	1
Next higher mode (TE)	Approximately $\pi(r_1 + r_2)$		$\frac{\lambda_0}{\pi(r_1 + r_2)}$	$\sqrt{1 - \frac{\lambda_0}{\pi(r_1 + r_2)}}$

a is width of rectangular guide (normal to E field), in meters.
b is height of rectangular guide (parallel to E field), in meters.
d is diameter of circular guide, in meters.
r_1 is outer radius of inner conductor, coaxial line, in meters.
r_2 is inner radius of outer conductor, coaxial line, in meters.
λ_0 is free-space wavelength at operating frequency $= \frac{3 \times 10^8}{f}$ meters.

that waves would appear to have if the electromagnetic field were visible. Group velocity is the velocity at which a pulse of energy, or a modulation envelope, is propagated. It appears strange, on first thought, that waves can appear to go faster than does the energy they convey, but this phenomenon can be observed by dropping a stone into any pond; the disturbance spreads as a ring or group of waves, and the velocity of the waves is faster than the velocity of the group. Waves on the water are continually originating at the inside of the ring; the individual waves travel faster than the group, and thus travel through the ring and vanish at its leading edge.

Mathematically, phase velocity is

$$v_\varphi = \frac{\omega}{\beta} \tag{13-26}$$

Group velocity is

$$v_g = \frac{d\omega}{d\beta} \tag{13-27}$$

On a lossless transmission line, phase and group velocity are equal. They are different, however, for any kind of wave propagation for which the velocity varies with frequency.

In a wave guide, velocity varies with frequency, and group velocity is computed as follows, using the formulas of Table 13-1:

$$\beta = \beta_0 G = \frac{\omega}{v_0}\sqrt{1 - \frac{f_c^2}{f^2}} = \frac{1}{v_0}\sqrt{\omega^2 - \omega_c^2} \tag{13-28}$$

$$\beta^2 v_0^2 = \omega^2 - \omega_c^2 \tag{13-29}$$

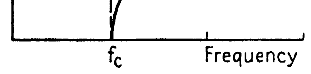

FIG. 13-7. Velocities in a wave guide.

Differentiating,

$$2\beta v_0^2\, d\beta = 2\omega\, d\omega \tag{13-30}$$

from which

$$v_g = \frac{d\omega}{d\beta} = \frac{\beta v_0^2}{\omega} = v_0 G \tag{13-31}$$

as in Table 13-1. Note that group velocity is slower than the speed of an unguided wave in free space, or a wave on a coaxial line, in the same proportion that phase velocity is faster (see Fig. 13-7). As cutoff is approached, the phase velocity (and the wavelength in the guide) become great without limit, while the group velocity diminishes toward zero. This means that a pulse of energy or a signal is propagated slowly.

If a steady, unmodulated wave of a single frequency is transmitted, only phase velocity can be observed. Group velocity has no meaning. No signal can be conveyed by such transmission. Energy enters one end of the guide and leaves the other end, but no velocity can be assigned to it.[5]

Phase velocity is, in a sense, an illusion. Nothing actually travels at phase velocity; certainly nothing material, and, indeed, not even energy. Thus there is no conflict with the theory of relativity. Yet the phase velocity of a wave in a guide is a physical reality. It is as real as the speed of a water wave on lake or ocean.

9. Reactive Attenuation. At any frequency higher than the cutoff frequency, γ in Eq. 13-5 is imaginary for a lossless guide. Since $\gamma = \alpha + j\beta$, α is zero at any frequency that makes γ imaginary, and β (as in Eq. 13-28) has the value

$$\beta = \frac{2\pi f}{v_0}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{13-32}$$

For a frequency lower than the cutoff frequency, γ is real; β then is zero, and α is

$$\alpha = \frac{2\pi f}{v_0}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} = \frac{2\pi}{\lambda_c}\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{13-33}$$

Equation 13-4 gives the distribution of voltage (or, more properly, the distribution of electric field strength) along the guide as

$$V = V_1 e^{\alpha x} \tag{13-34}$$

for frequencies below cutoff, α being a real number as in Eq. 13-33. Because α is real, there is no wave propagation of such frequencies. If a signal below cutoff frequency is introduced into a guide, there will be no wave. Electric and magnetic fields will penetrate only a limited distance into the guide. The electric and magnetic fields will die away, exponentially, with distance along the guide. The fields will not even resemble an attenuated wave, for they will not have the progressive phase variation that is characteristic of wave propagation; the entire field inside the wave guide, as far as it penetrates, will be in phase, and will grow strong and weak, each half cycle, at all points simultaneously.

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[5] Further discussion of group velocity, of Eq. 13-27, and of the application to water waves and other physical examples, is given in "Fundamentals of Electric Waves," Chap. XIV (Ref. 29). In this reference, phase velocity is compared to the speed of little waves that ripple along the back of a hurrying caterpillar, while the caterpillar himself travels at group velocity.

H. G. Booker (Ref. 3) compares waves to a series of railroad trains following one another along a main-line track; cars are being detached one by one from the tail of each train, to be attached on the front of the following train when it arrives. This peculiar process (analogous to the temporary storage of energy in the longitudinal field of the guide) makes each individual train appear to the uncritical observer to be traveling faster than any individual car is actually moving, and yet a shipment of goods on the line would unquestionably be delayed. The analogy is excellent.

The decrease of field strength with distance along the guide, for frequencies below cutoff, is called attenuation, but it is not the result of energy being lost in resistance of the guide walls. It would occur in an ideally lossless guide. It is owing to rejection of energy by the guide, not to waste of energy being propagated. It is another example of the *reactive attenuation* that was found in loaded lines and filter networks. The wave guide, as previously mentioned, is a high-pass filter.

Figure 13-6 may be interpreted as a plot of reactive attenuation as a function of frequency. The value of the radical in Eq. 13-33 can be read as an ordinate of the figure, instead of G, if f/f_c is used for the abscissa, instead of f_c/f. Since α is proportional to the radical, the curve is then a plot of attenuation below cutoff. It shows that reactive attenuation is high at frequencies substantially below cutoff, and approaches zero as cutoff frequency is approached.

A convenient and practical attenuator for high-frequency signals is made by passing the signals through a length of guide too small for wave propagation. The length of the attenuating section of guide is arranged to be variable (usually by making the pickup element or outlet of the guide movable) and thus the attenuation of the signal can be adjusted. Such a wave-guide attenuator provides a simple and reliable way of controlling output signal strength. It gives an amount of attenuation that varies exponentially with the adjustment of length, and that is substantially independent of frequency if the frequencies being attenuated are well below cutoff.

10. Resistive Attenuation. Only an ideally lossless wave guide would be able to transmit waves, even at frequencies in the transmission band, with no attenuation whatever. When a wave travels through the guide, current flows in the metal of the guide walls. Some power is necessarily wasted by this current. If the dielectric material within the guide is imperfect, there is also dielectric loss; this is important with solid or liquid dielectrics, but in air- or gas-filled guides it may be entirely neglected. The resistance loss, however, is always present, and in practical installations of long wave guides, as much as 20 per cent of the input power is sometimes wasted.

Information about the attenuation resulting from resistance in a rectangular guide can be obtained by including resistance in differential equations 13-1 and 13-2. If it is assumed that the inductance l of the top and bottom of the guide of Fig. 13-4 is accompanied by resistance r, and the inductance l' of the side walls is accompanied by the resistance r', these equations become

$$\frac{dV}{dx} = (r + j\omega l)I \tag{13-35}$$

$$\frac{dI}{dx} = \left(j\omega c + \frac{1}{r' + j\omega l'}\right)V \tag{13-36}$$

On differentiation, these yield

$$\frac{d^2V}{dx^2} = (r + j\omega l)\left(j\omega c + \frac{1}{r' + j\omega l'}\right)V \tag{13-37}$$

The propagation constant γ is the square root of the coefficient of V, and we may write

$$\gamma = \alpha + j\beta = \sqrt{(r + j\omega l)\left(j\omega c + \frac{1}{r' + j\omega l'}\right)} \tag{13-38}$$

The quantity under the radical is complex, the sum of a real term and an imaginary term. The square root of this sum is expanded by the binomial theorem, giving an infinite series. The real terms of the series are collected as an expression for α (the imaginary terms are $j\beta$). Assuming that ω is large, corresponding to high frequency, and that both r and r' are much smaller than either ωl or $\omega l'$, it is a good approximation to neglect all terms for α except the following:

$$\alpha = \frac{r}{2\sqrt{l/c}}\left[\frac{1}{G} + \left(\frac{lr'}{l'r} - 1\right)\left(\frac{1}{G} - G\right)\right] \tag{13-39}$$

G has the same meaning as in Eqs. 13-19, 13-20, or 13-21.

This expression for attenuation we have obtained from the equations of a transmission line with inductive shunt loading. Since (as in Fig. 13-4*b*) such a line is similar to a rectangular wave guide, it is to be expected that a similar expression will give attenuation in a wave guide. This is, indeed, true; wave-guide attenuation[6] is given by an expression of identical form, but written in terms of wave-guide dimensions, as

$$\alpha = \frac{R_s}{b\sqrt{\mu/\epsilon}}\left[\frac{1}{G} + \frac{2b}{a}\left(\frac{1}{G} - G\right)\right] \tag{13-40}$$

Attenuation is high near cutoff frequency, and also at very high frequency. G is zero at cutoff frequency, making attenuation infinite (this corresponds to the frequency that gives resonance in the shunt admittance of Fig. 13-5). At frequencies slightly above cutoff, attenuation drops rapidly (getting away from resonance in the shunt admittance). At high frequency, as G approaches 1, the bracket in Eq. 13-40 (or 13-39) approaches 1, and attenuation becomes proportional to resistance. Since high-frequency resistance is proportional to the square root of frequency, attenuation at frequencies several times cutoff is also proportional to the square root of frequency. These relations are shown in Fig. 13-8.

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[6] The derivation of Eq. 13-40, requiring use of electromagnetic theory, is not given here. See, for instance, "Fundamentals of Electric Waves," p. 223 (Ref. 29); or Ramo and Whinnery, "Fields and Waves in Modern Radio," Art. 9.04 (Ref. 23).

For computing from Eq. 13-40, a and b are dimensions in meters, and G is obtained from Table 13-2; μ and ϵ refer to the dielectric substance within the guide, and for air the value of $\sqrt{\mu/\epsilon}$ is 377 ohms. R_s is the "surface resistance" of the metal of the wave guide.

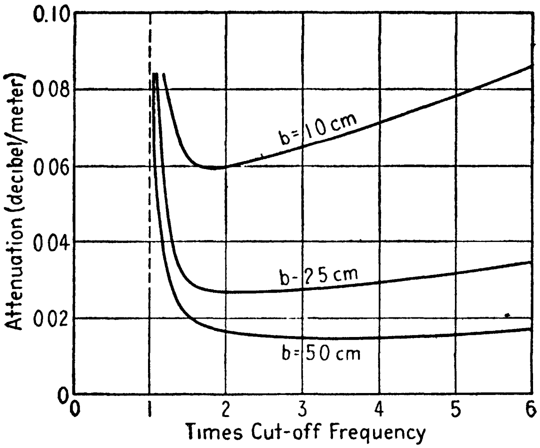

FIG. 13-8. Attenuation in a rectangular guide of copper, width a = 5 centimeters. (The dominant $TE_{1,0}$ mode is assumed.)

R_s is a resistance applicable under conditions of extreme skin effect. It is independent of the thickness of the metal, for substantially all the current is in a "skin" of very shallow depth at the metal surface. The skin-effect resistance of a metal surface one unit square is given by Eq. 7-43 as $R_s = \sqrt{\pi f \mu \rho}$. Introducing numerical values for copper, R_s is 2.61 $\sqrt{f}\ 10^{-7}$; for silver, R_s is 2.53 $\sqrt{f}\ 10^{-7}$. R_s is often called the surface resistance *per square*, for it is the resistance between two opposite edges of a square piece of the metal, and the value of this resistance is independent of the size of the square.

Figure 13-8 shows curves of attenuation for a rectangular copper guide 5 centimeters wide. As a guide of this width is usually about 2.5 centimeters in the other dimension, the middle curve of the three shown may be considered typical. These curves are computed from Eq. 13-40; to obtain attenuation in decibels per meter, rather than nepers, values from Eq. 13-40 are multiplied by 8.686.

11. Modes of Transmission. The electromagnetic pattern shown in Fig. 13-4*c* is only one of many that can be propagated in a rectangular wave guide, but it is much the most important because of having the lowest cutoff frequency. It is known as the dominant mode. Its electric field is purely transverse, and its magnetic field has an axial component.

The wave shown in Fig. 13-1*c* in a circular guide is not the dominant circular mode; the dominant circular mode is rather similar to the dominant rectangular mode. The mode shown is important because it has radial symmetry. Note that the mode in Fig. 13-1*c* has an axial component of electric field and a purely transverse magnetic field.

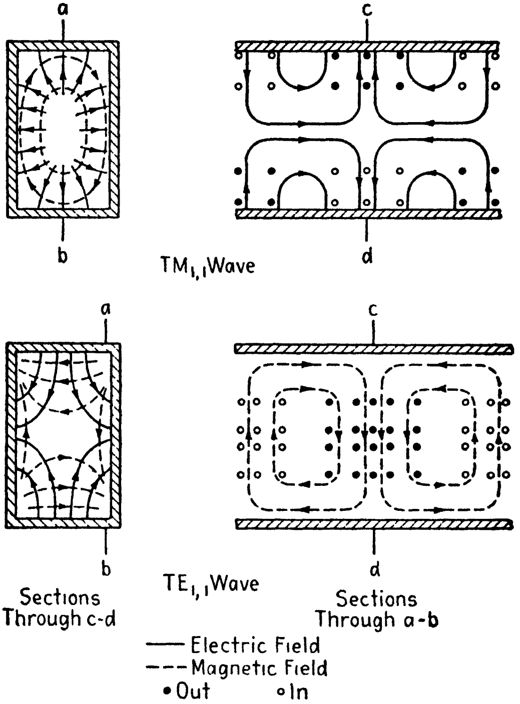

FIG. 13-9. Two higher rectangular modes. (*After Ramo and Whinnery.*)

All modes are divided into three classes: those having purely transverse electric fields (TE modes), those having purely transverse magnetic fields (TM modes), and those having electric and magnetic fields that are both purely transverse (TEM modes). TE and TM modes may exist together in a guide, producing complex patterns. TEM modes do not exist in hollow guides, but the ordinary transmisson-line mode is of this type.

Two rectangular modes other than the dominant mode of Fig. 13-4*c* are shown in Fig. 13-9. These three are the basic modes, and other rectangular modes are patterns in which one of these three basic modes is repeated in the space within the guide. Numerical subscripts (as for the $TE_{2,1}$ mode) indicate how many times the field pattern is repeated along each dimension of the cross section.

Similarly, many other circular modes are possible.[7] These have less practical importance than the dominant rectangular mode or the symmetrical circular mode.

It is interesting to discover that a coaxial line can propagate modes other than the usual TEM or transmission-line mode. Both TE and TM modes are possible in a coaxial line. They can exist, however, only at frequencies so high that the circumference of the coaxial line is of the same order of magnitude as the wavelength of the signal. These higher order modes, undesirable because they waste energy, are usually suppressed by designing the line to be too small to propagate them (see Sec. 1, Chap. 12).

The Single-wire Line. It is interesting that a single wire can be used as a wave guide, with a field pattern existing in the space about the one wire. The electric field does not extend to any other conductor, but returns to the same wire from which it issued; it is not unlike the pattern of Fig. 13-1*c* turned inside out. This mode is possible only if the phase velocity on the wire is less than the speed of light. Among the ways in which a wave can be slowed to meet this requirement are by the use of a conductor with a grooved surface (grooves increase the inductive effect) or the use of a metal conductor enclosed within a layer of dielectric material. Enameled wire has been used for the purpose.

The cost of a wave guide consisting of a single wire is, of course, low. Attenuation and radiation need not be excessive. An outstanding disadvantage is that there must be a large clear space about the wire to avoid interference with the fields of the wave.

12. Guide Impedance. Impedance is an important concept for wave guides, as it is for transmission lines. It enters into computations regarding termination of guides, and the effect of discontinuities. Traveling waves in guides are reflected at the guide terminals and at intermediate discontinuities. Incident and reflected waves combine to give standing waves. Reflection factor and standing-wave ratio relate to impedance in guides as they do in lines, and the same impedance charts are used.

The *normalized* impedance ζ of a transmission line was defined as Z/Z_0, the actual impedance divided by the characteristic impedance of the line.

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[7] For diagrams, see Refs. 6, 20, 23. 29, 33, etc.

For hollow guides, the normalized impedance ζ is used, also. Thus, if a guide is so terminated that there is no reflection, $\zeta = 1$. If the guide is short-circuited, giving a reflected wave equal in magnitude to the incident wave, $\zeta = 0$. In general, ζ is related to the reflection factor ρ, as in Eq. 12-88, by

$$\zeta = \frac{1 + \rho}{1 - \rho} \tag{13-41}$$

and ζ can have any complex value.

To determine an actual impedance in ohms is, for a hollow guide, more difficult. The impedance of a two-conductor line is the ratio of voltage to current, but for a hollow guide there is no general agreement on how to assign meaning to either voltage or current.[8] However, impedance of a transmission line may be conceived as the ratio of the electric field of the line to the magnetic field, as well as the ratio of voltage to current; voltage is the integral of electric field and current is the integral of magnetic field (Eqs. 6-71 and 6-1). This concept can be extended to hollow guides. It leads to the *wave impedance* or *field impedance* of a line or guide.

Wave impedance or *field impedance* is, by definition, the ratio of the transverse component of electric field to the transverse component of magnetic field. The characteristic wave impedance is the special value of the wave impedance that exists when the guide is carrying a single pure traveling wave with no reflected wave.

For a transmission line, having only transverse components of field, the characteristic wave impedance is equal to $\sqrt{\mu/\epsilon}$, μ and ϵ being the permeability and dielectric constant of the insulation. (This is also called the *intrinsic impedance* of the medium.) The numerical value of the characteristic wave impedance of any air-insulated transmission line is 376.7 ohms.

For a hollow guide having an axial component of magnetic field (any TE mode), as in Fig. 13-4, the characteristic wave impedance is $\sqrt{\mu/\epsilon}/G$. For a hollow guide having an axial component of electric field (any TM mode), as in Fig. 13-1, the characteristic wave impedance is $\sqrt{\mu/\epsilon}\,G$. These relations can be derived from the foregoing equations of this chapter, but the derivation will not be given as we shall make only passing reference to the wave impedance of lines and guides.[9]

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[8] Various possible definitions of impedance in hollow guides are nicely summarized in "Microwave Transmission Circuits," p. 52 (Ref. 22).

[9] The concept of impedance in the theory of wave propagation was suggested by Schelkunoff in the *Bell System Technical Journal* in 1938, and is expanded in his book (Ref. 27).

An excellent development of many aspects of wave propagation theory from simple transmission-line theory is given by Booker in his article in the *Journal of the Institute of Electrical Engineers*, 1947 (Ref. 3). Booker's article is particularly recommended to supplement this chapter.

For lines and guides, the field impedance divided by the characteristic field impedance is the normalized impedance ζ.

13. Termination of Wave Guides. *Short Circuit.* A coaxial line or a wave guide is short-circuited if the space through which the wave passes is completely closed by a sheet of metal. Thus, in Fig. 13-10*a*, a disc is shown short-circuiting a line. If such a disc is perfectly conducting (or, for practical purposes, if it is made of metal of good conductivity), it provides a perfect short circuit, and ζ at the disc is zero. No energy can be transmitted past such a short circuit.

The short-circuiting wire or strap shown between conductors in Fig. 13-10*b*, on the contrary, does not act as a complete barrier to electromagnetic waves of high frequency. Unless the diameter of the guide is very much smaller than the transmitted wavelength, there will be a significant amount of field strength in the region beyond the short-circuiting wire. Energy will pass beyond the wire. Indeed, the short-circuiting wire may be thought of as a small antenna that radiates energy. The impedance ζ at such a wire is low, but it is not zero. Transmission of energy beyond the wire gives ζ a resistive component, and temporary storage of energy in local electromagnetic fields gives ζ a reactive component.

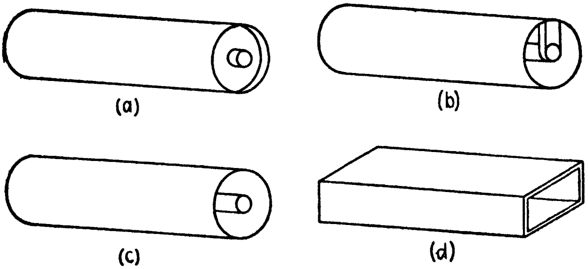

FIG. 13-10. Line and guide terminations.

Open Circuit. At very high frequency, the open coaxial line of Fig. 13-10*c* and the open guide of Fig. 13-10*d* do not provide open circuits in the sense that $\zeta = \infty$. Although a low-frequency wave must stop when it comes to the end of the line, a high-frequency wave is only partially stopped; some of its energy radiates into space. Radiation from an open line is quite negligible if the wavelength is much greater than the line dimensions, but at sufficiently high frequency it is not negligible, and it is never negligible from an open hollow guide. The value of ζ at the open end of a line or guide is high, but not infinite.

Resistive Termination. It is not usually satisfactory, at microwave frequencies, to terminate a line with a lumped resistor. If the connection from the inner to the outer conductor of Fig. 13-10*b* is a resistor, it will receive a certain amount of energy from an incident wave, but it will so distort the wave that some energy will pass beyond it, and some will be stored reactively in a local electromagnetic field. Although the connection is purely resistive, its effect on the field makes it equivalent to an admittance shunted across the line with properties of both conductance and susceptance, a complex admittance.

The ideal resistive load on a coaxial line is a disc of uniformly resistive material. If the disc closing the line of Fig. 13-10*a* is resistive, it will absorb energy without distorting the field. The amount of energy absorbed is dependent on the resistance of the disc.

The resistance of conducting sheet material is measured *per square*. By resistance *per square* is meant the resistance between opposite edges of a square piece of the sheet material, a value that is independent of the size of the square.

If the resistance of the material is 377 ohms per square, a disc of the material will terminate any air-insulated coaxial transmission line in its characteristic impedance. It is rather remarkable that such a disc will have the correct resistance regardless of the dimensions of the line. In Fig. 13-10*a*, for instance, the line is terminated in a resistor consisting of a round piece of material with a hole in the middle. If the resistance per square of the piece is 377 ohms, the resistance of the disc from the inner conductor to the outer conductor can be shown (by integration) to be equal to the characteristic impedance of the line. This is related, of course, to the fact that 377 ohms is the wave impedance of the coaxial line (see Sec. 12), and is the intrinsic impedance of air.

Material with 377 ohms resistance per square will also terminate a two-wire line in its characteristic impedance, if the dielectric medium is air, but since the fields about a two-wire line are unbounded, the terminating sheet would theoretically have to be of infinite extent.

The analogous type of resistive termination for a wave guide is a sheet of material that completely closes the guide. A rectangular section, for instance, is needed to cover the end of the guide of Fig. 13-10*d*. The correct resistance of the material for an air-filled guide is $377/G$ ohms for the usual mode in a rectangular guide (the wave of Fig. 13-4*c*) or for any other TE mode. It is $377G$ ohms for the wave of Fig. 13-1*c*, or for any other TM mode. (G is as in Table 13-2 or Fig. 13-6.) These are the values of wave impedance of the guides.

Now, however, it is necessary to take into account an aspect of centimeter-wave transmission that does not need to be considered at lower frequencies. A resistive curtain across a line or wave guide will not totally block the flow of energy. Current in the curtain itself will produce electromagnetic field in the space beyond the curtain.

Let us think first of a low-frequency line. A two-wire line is shown in Fig. 13-11 for convenience. Resistance may be connected at the end, as in Fig. 13-11*a*, or at a mid-point, as in Fig. 13-11*b*. In Fig. 13-11*a*, the line is terminated in its characteristic impedance, and all the energy of an incident wave is absorbed in the resistance. In Fig. 13-11*b*, energy is not all absorbed, for the first section of line is terminated by the combination of the resistance Z_0 in parallel with the input impedance to the continuing section of line.

If it is desired to have the energy of an incident wave completely absorbed in the resistance Z_0 even though the line continues on, this may be done by short-circuiting the line at a distance of $\lambda/4$ beyond the point of attachment of the resistance (see Fig. 13-11*c*). The impedance of the short-circuited section of line, looking toward the short circuit from the point of connection of the load, is infinite. Thus the first section of line is provided with a terminating impedance of Z_0 in parallel with an infinite impedance, the total impedance being, of course, Z_0, as desired.

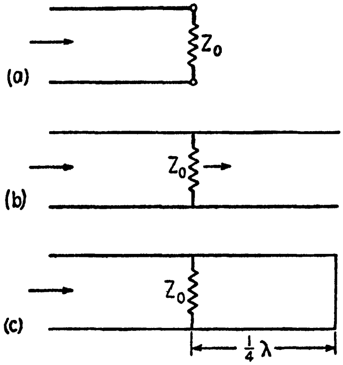

FIG. 13-11. Termination for complete energy absorption.

This principle is used for a line or guide carrying high-frequency waves, of wavelength comparable to the cross-sectional dimensions of the line or guide. At low frequency, a termination of this kind is necessary only if the line continues on beyond the point of attachment of the load. At sufficiently high frequency, however, energy passes on beyond the load even if the line does not continue, for if the line does not continue, energy radiates into space. Transmission of energy past the load resistor is then prevented by extending the line an additional quarter-wavelength, and there short-circuiting it.

Since a hollow guide can operate only at frequencies for which the wavelength is comparable to the guide dimensions, it never acts like the line of Fig. 13-11*a*, but always like that of Fig. 13-11*b*. A hollow guide must always be terminated with a quarter-wavelength short-circuited section, as in Fig. 13-11*c*, if the energy is to be kept under control.

Irregularities. Wherever the regularity of field pattern in a line or guide is deformed, as at loads, bends, junctions, supports, or probes, there is a disturbance of the relation that normally exists between energy stored in the electric field and energy stored in the magnetic field. This effect, which was mentioned in Sec. 22, Chap. 12, is equivalent to the addition of a susceptive shunt.

If the electric field is strengthened relative to the magnetic field, the irregularity is equivalent to a capacitive shunt. If the magnetic field is relatively more strengthened, it is equivalent to an inductive shunt.

Such equivalent shunt capacitances and inductances, being of the order of micro-microfarads or micro-microhenrys, are quite negligible at low frequencies. The energy stored in the distorted field is significant only if the dimensions of the irregularity producing the distortion are an appreciable fraction of a wavelength.

14. Reactive Obstacles. Since an irregularity in a wave guide has the effect of a susceptance shunted across the guide, a simple means of obtaining a desired susceptance is suggested. Among the common forms of obstacle used to introduce intentional susceptance are diaphragms or irises that partly close the guide, and rods, probes, or screws that project into the guide.

Figure 13-12*a* shows a rectangular diaphragm with inductive properties; the "window" is a vertical slot with edges parallel to the electric field. If the slot in the diaphragm is horizontal, as in Fig. 13-12*b*, the effective susceptance is capacitive.[10] The window of Fig. 13-12*c* can introduce either inductive or capacitive susceptance depending on the frequency; it acts like a parallel-resonant circuit and changes from inductive to capacitive susceptance at the resonant frequency. At the resonant frequency, the equivalent shunting susceptance is zero, and for that particular frequency the "window" is completely transparent although it may reflect a substantial part of the incident wave at higher and lower frequencies. To be highly frequency-selective, such a window should be narrow and about one-half wavelength long.

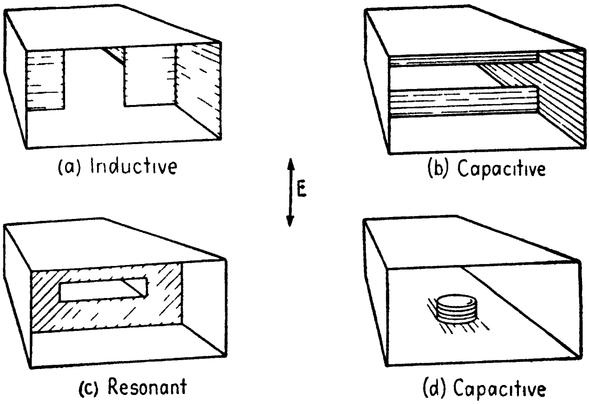

FIG. 13-12. Wave-guide susceptances.

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[10] Design formulas are available in various books. See, for instance, Refs. 6, 20, 22, 24, and 25.

The most convenient susceptance results from permitting a screw, as in Fig. 13-12*d*, to project into the guide. Such a screw, or other metal probe, if shorter than $\frac{1}{4}$ wavelength, introduces capacitive susceptance. A quarter-wavelength probe has the effect of a series-resonant shunt and produces almost complete reflection of an incident wave. If the probe is longer (or the frequency higher), its effect is inductive. (Note that a long, narrow piece of metal parallel to the E field has properties that are in general inverse to those of a long, narrow slot normal to the E field.)

A short screw, as in Fig. 13-12*d*, can be used for tuning. It serves the same purpose as a short-circuited stub on a transmission line, but more simply. If it is desired to eliminate a reflected wave that exists in a guide because of imperfect matching of impedance at a load, or because of some irregularity in the guide itself, a tuning screw can be introduced into the guide to produce an equal and opposite reflection. The reflected wave can thus be eliminated, and the guide operated *flat*, without standing waves.

The effective length of a tuning screw is easily adjustable, as it can be screwed in or out of the guide. If only one tuning screw were used, it would have to be located at the correct position along the guide, as well as being adjustable at that point. It is more practical to use two or even three tuning screws in fixed positions. The problem is strictly analogous to that of eliminating reflections on a transmission line by use of one, two, or three stub lines (see Secs. 18 to 20, Chap. 12).

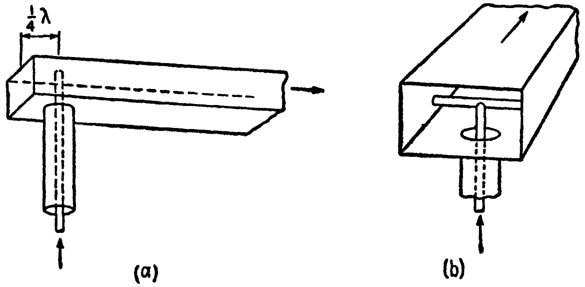
15. Coupling to Guides. We have not yet discussed how energy is to be fed into a wave guide, or how the guide can deliver useful energy to a load. Any means used to insert energy can, by the reciprocity theorem, equally well be used to extract it; thus it will only be necessary to consider means of coupling to guides without specifying whether the coupling is to be used for transmitting or receiving energy.

A wave guide can be coupled to free space, or to a cavity. Energy will come out of a guide, as previously mentioned, if the end is merely left open. Similarly, energy will go into a guide if the open end is in a region (as in a cavity resonator) where an electromagnetic field of sufficiently high frequency exists. Radiation from the open end of a guide can be increased and directed by flaring the guide into a horn, and practical antennas are so constructed. [See Terman's "Radio Engineering," Chap. 14 (Ref. 33).] A slot in the wall of a wave guide will also radiate energy.

It is often necessary to couple a wave guide to a coaxial line. The obvious connection, if a rectangular guide is considered to be related to a parallel-strip line, is to attach one conductor of the coaxial line to the bottom of the guide and one to the top. This may be done as in Fig. 13-13*a*. The left-hand end of the wave guide is closed by a short-circuiting plate placed $\frac{1}{4}$ wavelength from the point of connection, to prevent transmission or radiation in the wrong direction, and energy from the line flows along the guide toward the right.

This type of coupling has the disadvantage of being unadjustable. It is easier to eliminate reflected waves in the line and guide if the inner conductor of the coaxial line is terminated in a probe that projects into the guide but does not make contact with the guide wall. Figure 13-13*a* may be used to illustrate this device if it is understood that the inner line conductor does not touch the top wall of the guide. Adjustment is obtained by varying the depth of penetration of the probe into the guide. If the position of the probe can also be adjusted, across and along the guide, there are three degrees of freedom available to assist in tuning.

The coupling probe may be looked upon as a small antenna, radiating into the wave guide or receiving energy from the guide. It may be given a cross piece, equivalent to a "flat-top" antenna, to increase the coupling, as in Fig. 13-13*b*. (The short-circuiting end plate which is required for efficient operation is omitted in this figure.)



(a) (b)

FIG. 13-13. Coupling between coaxial line and wave guide.

The equivalent of a loop antenna may be obtained by bending the inner conductor of the coaxial line into a loop that is attached to the outer conductor of the coaxial line. The loop is then inserted into the wave guide. This connection is simple and rugged, but unadjustable.

Whatever the method of coupling used, the electric and magnetic fields about the end of the coaxial line must have the same general form and orientation as the fields it is desired to excite in the wave guide. The greater the similarity, the greater will be the energy transfer. It would, of course, be desirable to provide a coupling device that would terminate the coaxial line in its characteristic impedance with no reactive component ($\zeta = 1$ for the line) and that would at the same time terminate the wave guide in such a way that no standing waves result ($\zeta = 1$ for the guide, also), and this condition can be approached by careful design. Ordinarily, however, it is necessary to complete the tuning by means of adjustments that can be made after assembly of the apparatus. Thus, tuning screws can be used to eliminate standing waves in the guide, the standing-wave ratio being measured experimentally while the tuning is being done.

Evanescent Field Forms. At the point of coupling to a wave guide, or at any irregularity or discontinuity, the electric and magnetic fields do not have the simple form of a single mode shown in Figs. 13-1*c*, 13-4*c*, or 13-9. The fields in the guides of Fig. 13-13, for instance, are formed largely by the coupling stubs and near the coupling they have only a remote resemblance to the desired mode. Fields in the guides of Fig. 13-12 are likewise shaped to conform to the obstacles in the guides.

However, any possible field pattern can be analyzed into modes of the kind already considered. The analysis is analogous to Fourier analysis of waves into sinusoidal components. The total complicated field pattern about a coupling probe, for instance, results from the superposition of a fundamental mode and higher order modes of appropriate phase and magnitude.

The fundamental mode is propagated along the wave guide. Modes of high order are not propagated; the cutoff frequency of a high-order mode is too great, and it cannot travel as a wave. It dies away in a short distance along the guide. The complicated field that exists about the coupling or other irregularity is therefore localized. Only component modes of sufficiently low order can propagate. The high-order components which cannot propagate along the guide are called *evanescent forms.*

It is usually desirable so to design a guide that only the fundamental or dominant mode will propagate, and the higher order modes will all be evanescent or local. This is done by proper selection of the guide dimensions, as explained in the next section.

16. Practical Considerations. In actual practice, the most common wave guide is rectangular, with the greater dimension of the cross section twice the lesser dimension. The electromagnetic field pattern of Fig. 13-4*c* (the TE$_{1,0}$ mode) is used. The size of the guide is selected to make the operating frequency about 1.5 times the cutoff frequency; that is, the width of the guide is about 1.5 times the free-space half-wavelength of the signal.

The operating frequency must be great enough to avoid the high attenuation that occurs at frequencies near cutoff (Fig. 13-8); yet the frequency must not be so great that undesired modes can propagate in the guide. If the width of the guide is twice its height, the cutoff frequency of undesired modes is two or more times the cutoff frequency of the desired mode (Table 13-2), and if the operating frequency is 1.5 times cutoff for the dominant mode, it is still well below cutoff for other modes.[11]

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[11] For such a guide, there are two higher modes with the same cutoff frequency. One (the TE$_{0,1}$) is identical with the desired mode, but with its electric field parallel

The ratio of two to one in transverse dimensions gives definite control of the polarization of the transmitted wave. That is, the electric field can only be parallel to the lesser dimension of the cross section. If the lesser dimension were much increased, this control would be lost. If, on the other hand, the dimension were much decreased, attenuation would be considerably greater. It is these considerations that dictate the optimum proportions.

Cylindrical guides are occasionally used. There is no control of polarization in a round guide, and this is a disadvantage in most applications. Also, there is less difference between the cutoff frequencies of various modes, and no operating frequency will give reasonably low attenuation of the dominant mode and, at the same time, high attenuation of all higher modes. Circular guides, however, permit the use of rotating mechanical joints, and this is sometimes necessary; a moving antenna, for example, may be fed through a circular guide. By using a mode with radial symmetry (the $TM_{0,1}$ mode), problems of polarization are avoided.

The practical choice between use of a hollow guide or a coaxial line is largely determined by frequency. If the free-space wavelength of the signal is greater than 20 or 30 centimeters, a hollow guide must be 15 or 20 centimeters broad, and this is inconveniently bulky. At the other extreme, construction of a coaxial line is troublesome if the free-space wavelength is less than 2 or 3 centimeters. To prevent modes other than the ordinary transmission-line mode from being propagated on a coaxial line, the free-space wavelength of the signal must exceed the average of the circumferences of the inner and outer conductors, and this means that if the wavelength is 3 centimeters the inside diameter of the outer conductor must be less than 1 centimeter.

Attenuation is less in a hollow guide than in a coaxial line of similar over-all dimensions. Power loss in the coaxial line, including the effect of supports, is roughly three times that in a hollow guide.

The maximum power that can be transmitted by line or guide is limited by sparking, for dielectric breakdown results if the electric field becomes too great. The capacity of a hollow guide is several times that of a comparable coaxial line.

For these various reasons, hollow guides are commonly used for signals with free-space wavelengths less than about 10 centimeters, and transmission lines are usually used for greater wavelengths.

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to the longer dimension of the guide instead of parallel to the shorter dimension. The other (the $TE_{2,0}$) may be described as consisting of two identical waves, each of the dominant mode pattern, traveling along the guide side by side. Both of these modes can exist if the operating frequency is more than twice the cutoff frequency of the dominant mode.

PROBLEMS

13-1. Show that the network of Fig. 13-5 is a band-pass filter if its elements are of finite size and that it becomes a high-pass filter if its series reactance and shunt admittance both become small without limit. (An infinite series of such infinitesimal sections thus represents a wave guide.)

13-2. Figure 13-4*b* is a diagram of a transmission line *loaded* with inductive shunts. Show that a *shunt-loaded* transmisson line acts as a high-pass filter provided the distance between *loads* is less than approximately one-half wavelength. (Thus a wave guide, in which *loading* is continuous, will pass all high frequencies.)

13-3. Compute the factor G for hollow rectangular wave guides (transmitting the dominant mode) as follows:

	Frequency, megacycles	Width a, centimeters	Height b, centimeters
a.	3,000	6.0	3.0
b.	5,000	6.0	4.0
c.	5,000	5.0	2.5
d.	3,100	5.0	2.5

13-4. Compute the factor G for hollow rectangular wave guides (transmitting the dominant mode) as follows:

	Frequency, megacycles	Width a, centimeters	Height b, centimeters
a.	6,000	5.0	2.5
b.	4,500	5.0	3.0
c.	4,500	3.5	1.5
d.	3,000	7.5	4.0

13-5. Find phase velocity, group velocity, and wavelength in the guide for each of the four guides of Prob. 13-3.

13-6. Find phase velocity, group velocity, and wavelength in the guide for each of the four guides of Prob. 13-4.

13-7. Show that reactive attenuation is given in decibels per unit length by $(54.6/\lambda_c)\sqrt{1-(\lambda_c/\lambda)^2}$. Also write an expression for resistive attenuation, similar to Eq. 13-40, but giving decibels per unit length specifically for a hollow copper guide. Put the expression in the simplest possible form for easy use.

13-8. A hollow copper wave guide is 5 by 2.5 centimeters. Compute and plot attenuation in decibels per meter at the following frequencies:

a. 10 per cent below cutoff.

b. 1 per cent below cutoff.

c. 1 per cent above cutoff.

d. 10 per cent above cutoff.

13-9. A high-frequency coaxial line is terminated by a resistive disc that occupies the annular space between the inner and outer conductors (Fig. 13-10*a*). Show (by integration) that the resistance from one conductor to the other is equal to Z_0 if the disc has a resistance of 377 ohms per square.

13-10. Each of the guides of Prob. 13-3 is to be terminated by a sheet of resistive material (followed by a short-circuited quarter-wavelength section of guide). Find the correct resistance per square of resistive sheet for each guide.

13-11. Find the correct resistance per square of resistive sheet to terminate (as in Prob. 13-10) each of the guides of Prob. 13-4.

13-12. Each of the guides of Prob. 13-3 is closed by a sheet of material with resistance of 377 ohms per square (followed by a short, circuited quarter-wavelength section of guide). What standing-wave ratio results in each guide from the incorrect termination?

13-13. Repeat Prob. 13-12 for each of the guides of Prob. 13-4.

CHAPTER 14

TRAVELING WAVES

1. Transient Phenomena. Previous chapters of this book have been concerned with the behavior of transmission lines when the voltages and currents are of a single alternating frequency, steadily applied. We now turn our attention to transient effects. The simplest transient response is obtained by the sudden application of a constant voltage, as when a battery is connected to a transmission line by closing a switch.

2. An Approximate Network. Let us consider that the inductance and capacitance of a transmission line, instead of being truly distributed, are gathered into small but finite lumps as in Fig. 14-1a. Resistance will, for the present, be neglected. A battery voltage E is suddenly applied to the sending end of the line by closing the switch. Current immediately begins to flow through inductance L_1 into condenser C_1. As C_1 becomes charged, so that there will be voltage across it, current starts to flow through L_2 into C_2. C_2 becomes charged, and current then flows through L_3 into C_3, and this action continues on down the line. First C_1 is charged to voltage E, then C_2 is charged to voltage E, then C_3 is charged to voltage E, and so on. There is a voltage wave traveling along the line; it starts at the sending end and travels at a certain velocity, and all of the line that it has reached is charged to voltage E, while the line beyond the front of the wave is yet uncharged. This is shown graphically in Fig. 14-1b, in which voltage is plotted as a function of distance along the line. The wave has traveled to the point x_1, so that the line to the left of x_1 is charged to voltage E, and the front of the wave is continuing toward the right as indicated by the arrow.

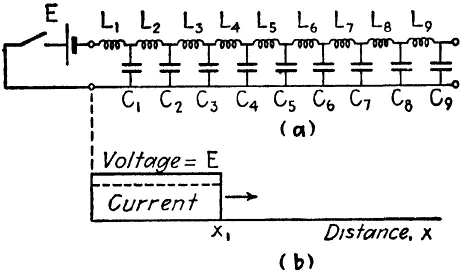

FIG. 14-1. A traveling wave.

The voltage wave is accompanied by a current wave. While C_1 is being charged, current flows through L_1. When C_1 becomes charged, current starts to flow through L_2, and the current to L_2 is supplied through L_1. Then current is extended into L_3 and continues to flow through L_1 and L_2. A current wave progresses along the line in this way, coincident with the voltage wave. Each is necessary to the other. In fact, the current and voltage are not separate waves; they are merely different aspects of the same wave of energy.

For easier visualization of the action of distributed parameters, this discussion has been based on a mental picture of a line in which inductance and capacitance are collected into small but finite lumps. But the discussion has been simplified by neglecting certain effects that are inseparable from lumped parameters. When voltage is applied to the artificial line of Fig. 14-1*a*, current will begin to flow in L_1, but gradually and not instantaneously. C_1 will charge gradually, and current will begin in L_2 as soon as there is any voltage across C_1, not waiting for C_1 to reach voltage E. Moreover, the voltage of C_1 will not rise abruptly to E and then continue at that value thereafter; instead, it will rise in a finite time and will increase to a value greater than E, finally approaching E as a limit. But all these differences from ideal behavior are due to "lumpiness" of the line, and become of less magnitude as the number of lumps is increased and the size of the individual lumps is decreased; and in the limiting case of the actual transmission line with truly distributed parameters, the differences disappear.

3. The Differential Equations. Although a mathematical solution can progress some distance by consideration of the "lumpy" line,[1] it is simpler to make use of the differential equations of the transmission line. When it is *not* assumed that voltage and current are sinusoidal, the differential equations are written in terms of instantaneous voltage and current.

Such equations were given in Chap. 1 (Eqs. 1-7 and 1-17). However, in traveling waves it is natural to measure distance x from the transmitting end of the line. Transient disturbances are studied as they travel from their source, as in Fig. 14-1. For steady-state phenomena, on the other hand, we found it necessary to begin investigations at the load. For convenience in this chapter, let us measure distance from the sending end of the line; this changes the sign of x, and with this change the differential equations from Chap. 1 are

$$\frac{\partial v}{\partial x} = -\left(ri + l\frac{\partial i}{\partial t}\right) \tag{14-1}$$

$$\frac{\partial i}{\partial x} = -\left(gv + c\frac{\partial v}{\partial t}\right) \tag{14-2}$$

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[1] As is done in "Transient Electric Currents," Chap. IX (Ref. 30). Much of the present chapter is there treated in greater detail.

These are partial differential equations, as they contain the two independent variables distance x and time t.

Simultaneous solution of these equations is possible, but not easy. However, if energy loss along the line is neglected, so that r and g are both zero, the solution is perfectly simple. Then

$$\frac{\partial v}{\partial x} = -l\frac{\partial i}{\partial t} \tag{14-3}$$

$$\frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \tag{14-4}$$

By differentiating Eq. 14-3,

$$\frac{\partial^2 v}{\partial x^2} = -l\frac{\partial}{\partial x}\left(\frac{\partial i}{\partial t}\right) = -l\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial x}\right) = lc\frac{\partial^2 v}{\partial t^2} \tag{14-5}$$

Thus, by substitution of Eq. 14-4 to eliminate i, we have obtained an equation entirely in v.

In a similar manner, v can be eliminated from Eq. 14-4 to give an equation in i only:

$$\frac{\partial^2 i}{\partial x^2} = lc\frac{\partial^2 i}{\partial t^2} \tag{14-6}$$

The solutions of these Eqs. 14-5 and 14-6, the so-called *wave equations*, must now be considered. In functional notation, the solutions are

$$v = v_1\left(x - \frac{1}{\sqrt{lc}}t\right) + v_2\left(x + \frac{1}{\sqrt{lc}}t\right) \tag{14-7}$$

$$i = i_1\left(x - \frac{1}{\sqrt{lc}}t\right) + i_2\left(x + \frac{1}{\sqrt{lc}}t\right) \tag{14-8}$$

Equation 14-7 can be expressed in words as follows: The voltage on a transmission line is the sum of two functions, v_1 and v_2, the first of which is a wave traveling in the positive x direction, and the second a wave traveling in the opposite or negative x direction. The speed of travel of both waves is the same:

$$\text{Speed} = \frac{1}{\sqrt{lc}} \tag{14-9}$$

It will be recognized that this is very nearly the speed of light.

In describing voltage as a function v_1 of $x - (1/\sqrt{lc})t$, Eq. 14-7 does not tell what function v_1 must be; it may be a function of any form: sinusoidal, exponential, algebraic, or other. It says that *any* function will satisfy the differential equation 14-5 if the variables x and t appear in it only in the combination $x - (1/\sqrt{lc})t$.

To show that this is correct, consider a voltage v_1 of the general functional form

$$v_1\left(x - \frac{1}{\sqrt{lc}}\,t\right) \tag{14-10}$$

Differentiation (according to the rules for partial differentiation of composite functions) shows that

$$\frac{\partial v_1}{\partial x} = -\sqrt{lc}\,\frac{\partial v_1}{\partial t} \tag{14-11}$$

and another differentiation gives

$$\frac{\partial^2 v_1}{\partial x^2} = lc\,\frac{\partial^2 v_1}{\partial t^2} \tag{14-12}$$

Since any function such as 14-10 gives this result, any such function satisfies Eq. 14-5.

It remains to be seen that a function of $x - (1/\sqrt{lc})t$ is a traveling wave. If voltage on a line is described by the function $v_1[x - (1/\sqrt{lc})t]$, and a constant value is assigned to t, the function becomes an expression of voltage distribution at a particular instant as a function of x. If, instead, a constant value is assigned to x, it describes voltage at a given point as a function of time. By assigning specific values to both x and t, the particular voltage at a fixed time and a fixed place results.

Finally, the voltage at a moving point may be investigated by properly relating x and t. Let us specify that a point is to move with velocity equal to $1/\sqrt{lc}$. Then, if x is the position of the point and t is time,

$$\text{Velocity} = \frac{dx}{dt} = \frac{1}{\sqrt{lc}} \tag{14-13}$$

from which

$$x - \frac{1}{\sqrt{lc}}\,t = \text{constant} \tag{14-14}$$

The traveling wave of voltage is described by a function of $x - (1/\sqrt{lc})t$, and such a function, by Eq. 14-14, reduces to a constant value at the moving point.

Physically, this means that if we observe voltage on a transmission line and keep our point of observation moving along the line at a speed of $1/\sqrt{lc}$, the voltage we observe remains unchanging. It is merely putting this idea into other words to say that the voltage on the line appears to be traveling with unchanging amplitude and form, the speed being $1/\sqrt{lc}$.

Thus the first term of Eq. 14-7 describes a voltage wave traveling in the positive x direction; similarly the second term describes a wave traveling in the negative x direction. Current, by Eq. 14-8, has similar components. These traveling waves are not attenuated or distorted as they travel because we are assuming a line without losses. In Chap. 1, steady-state phenomena were interpreted as traveling waves; the present discussion is a generalization to apply to waves of any form.

There has been nothing in this section to define the form of a traveling wave. Mathematically, it may be any shape. Physically, the shape of the wave depends on the voltage applied to the line. If the applied voltage is sinusoidal (as a function of time), the traveling wave is sinusoidal (as a function of distance); if the applied voltage is a square wave, the traveling wave is a square wave.

4. Characteristic Impedance. We know that voltage and current waves are not independent of each other. To study the relation, let us integrate both sides of Eq. 14-4:

$$\int \frac{\partial i}{\partial x}\,dx = -c\int \frac{\partial v}{\partial t}\,dx \tag{14-15}$$

In the right-hand member, take v_1 to be the traveling wave moving in the positive direction $v_1[x - (1/\sqrt{lc})t]$; then

$$\int \frac{\partial i_1}{\partial x}\,dx = -c\int \frac{\partial v_1}{\partial t}\,dx \tag{14-16}$$

The left-hand member integrates to give i_1, and applying the rules of partial differentiation to the right-hand member, from Eq. 14-11,

$$i_1 = \frac{c}{\sqrt{lc}}\int \frac{\partial v_1}{\partial x}\,dx = \sqrt{\frac{c}{l}}\,v_1 \tag{14-17}$$

or

$$\frac{v_1}{i_1} = \sqrt{\frac{l}{c}} = Z_0 \tag{14-18}$$

From this we learn that the relation of current to voltage on a lossless line is the same for a wave of any general form that it is for a steady-state sinusoidal wave: the ratio of voltage to current is the characteristic impedance. Z_0 is sometimes called *surge impedance* when speaking of transient waves.

The above discussion referred to a wave traveling in the positive x direction. The similar relation for a wave traveling in the negative x direction is

$$\frac{v_2}{i_2} = -\sqrt{\frac{l}{c}} = -Z_0 \tag{14-19}$$

These relations are strictly exact for a lossless line only, but are approximate for any ordinary line.

5. Reflection of Waves at Open End of Line. At the speed of light, it will not take long for a wave to reach the distant end of any actual line. Nearly 200 miles is traversed in 0.001 second. When a wave arrives at the end of the line its orderly progress is interrupted.

Figure 14-2 shows the action of a wave on arriving at the open-circuited end of a line—a smooth line, if the inductances and capacitances shown are considered to be infinitesimal. Figure 14-2*b* shows the flat-topped wave approaching the end of the line. The two parts of the energy wave, voltage and current, are progressing together; L_v is carrying current but L_z is not, and C_v is charged to voltage E but C_z is not. There is voltage, however, across C_v, so current will immediately begin to flow through L_z, and since L_z will carry as much current as L_v, the charge on C_v will increase no further. The voltage of C_v will therefore become and remain E. Current entering L_z will flow into the capacitance C_z, charging it to voltage E.

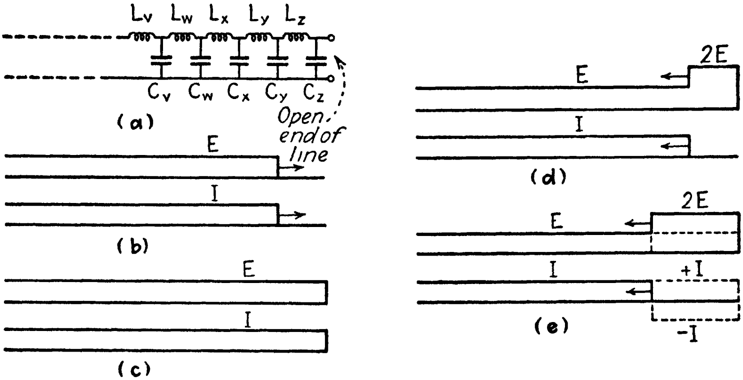

FIG. 14-2. A reflected wave.

The energy wave, at this instant, has reached the end of the line, as shown in Fig. 14-2*c*. All capacitance is now charged to voltage E, and all inductance is now carrying current I. But the procedure by which the wave has progressed along the line cannot be continued here: L_z is carrying current to C_z, and there is no inductance beyond C_z to act as an outlet for current as C_z becomes charged. The result is that C_z becomes overcharged, for L_z cannot stop carrying current until its magnetic energy is exhausted. L_z will continue to drive current into C_z, and, because of its magnetic energy, the current cannot be stopped until the voltage of C_z has been raised to double its normal value, or $2E$. When the voltage across the last element of capacitance is $2E$, current will stop flowing in the last element of inductance. This condition is illustrated in Fig. 14-2*d*; the voltage of C_z is now $2E$, and current in L_z has dropped to zero.

When L_z has ceased to carry current, all current carried by L_y is driven into C_y. This doubles the voltage of C_y, and current in L_y is forced to stop. At the same time, voltage is equalized at the two ends of L_z, and the overcharge on C_z cannot escape. When this has happened conditions are as shown in Fig. 14-2*e*: voltage is $2E$ across C_y and C_z, and current is zero in L_y and L_z. This procedure continues progressively along the line; from right to left the voltage becomes doubled and the current drops to zero.

It appears that when the original traveling wave reached the open end of the line, something of the nature of reflection took place. A traveling wave was reflected from the end and started to progress in the opposite direction. The reflected wave is shown by dash lines in Fig. 14-2*e*. It is a wave of voltage traveling toward the left, of magnitude E, which adds to the previously existing voltage. It makes the total voltage, as far as the reflected wave has progressed, $2E$. It is accompanied by a reflected wave of current that travels in the same direction at the same speed, but the reflected current wave is a negative wave, $-I$, and where it is added to the current of the original wave the total current becomes zero.

The front of the current wave as shown in Fig. 14-2*e* is progressing toward the left. It should be specially emphasized that this does not imply any reversal of flow of current. The current was originally flowing from the battery toward the open end of the line, and it still does so, after reflection, in that part of the line which continues to carry current. It is only the propagation of the wave front, and not the actual total current, that reverses direction when the wave is reflected. Current flows from left to right as the wave travels outward to charge the capacitance of the line to voltage E, and it continues to flow from left to right as the reflected wave returns, doubling the charge on the line as the voltage is raised to $2E$.

The physical meaning of a negative current is a current flowing from right to left, whereas a positive current flows from left to right by definition. If positive voltage is being built up by positive charge on a transmission-line wire, and it is being supplied by current flowing from the left, that will be a positive current. If the same charge is supplied by current from the right, the current will be negative. That is why a positive voltage wave traveling from the left is accompanied by a positive current, whereas the same wave traveling from the right requires a negative current.

When the wave of Fig. 14-2 reached the open end of its line, the boundary condition that determined the nature of the reflection was that, at the open end, there could be no current. At that point it must always be true that $i = 0$. From this fact we may find the reflected wave.

Equation 14-8 describes the current at all points on the line, including a point just at the open end where the current is zero. If the length of the line is D, and voltage and current are given the subscript r at the open end of the line (the receiving end), the current

$$i_r = i_1\left(D - \frac{1}{\sqrt{lc}}\,t\right) + i_2\left(D + \frac{1}{\sqrt{lc}}\,t\right) \tag{14-20}$$

Equation 14-20 is a function of time alone, for it describes the current at only one point: that is, at the receiving end of the line. It is made up of two parts; the first is due to the incident wave and the second to the reflected wave. Since the two terms must always add to zero,

$$i_2(D + vt) = -i_1(D - vt) \tag{14-21}$$

The two waves, in other words, must be equal and opposite at the terminating point of the line. The function i_2 must therefore describe the same shape of wave as the function i_1, but it must be of negative polarity, it must travel in the opposite direction, and it must leave the receiving end of the line at the instant of arrival of the incident wave.

The voltage of the reflected wave is related to the current of the reflected wave by Eq. 14-19; since the current of the reflected wave is opposite to the current of the incident wave, the voltage of the reflected wave is equal to the voltage of the incident wave. Total line voltage is the sum of the two waves, as in Eq. 14-7.

So, by mathematical reasoning, the results of Fig. 14-2 are substantiated

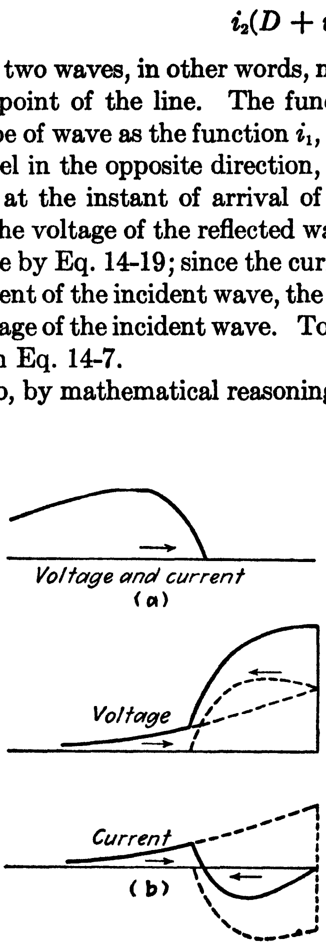

FIG. 14-3. Reflection at open-circuited end.

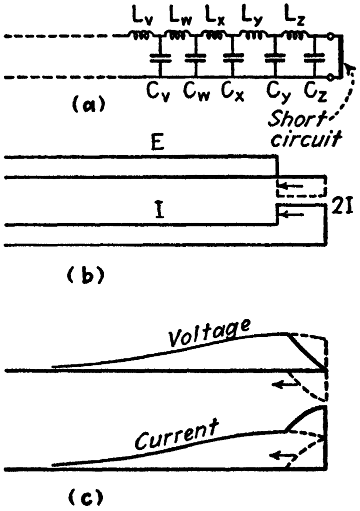

FIG. 14-4. Reflection at short-circuited end.

and are generalized to apply to waves of any shape. Figure 14-3 shows the result of reflection of a wave that is not flat. A wave is approaching the end of the line in Fig. 14-3*a*. A short time later, as shown in Fig. 14-3*b*, the front part of the wave has been reflected. The voltage wave is reflected as if it had been folded back on itself, and total voltage is the sum of the two waves; voltage at the end is always twice the voltage of either wave alone.[2] The current wave is reflected with reversal of sign, and the current at the end is always (as it must be) zero.

6. Reflection at Short Circuit. If the end of the line had been short-circuited instead of open, the action would have been entirely different. Figure 14-4*a* is a diagrammatic representation of a short-circuited line. As a traveling wave approaches the end, C_x becomes charged through L_x, and then L_y begins to carry current to C_y; C_y is charged to a voltage E, and L_z begins to carry current. But C_z can take no charge; it is short-circuited. Current will flow freely from L_z through the short circuit and into the return wire, like water from the open end of a pipe. So more and more current flows through L_z until the charge on C_y becomes exhausted. The current through L_z and through the short circuit becomes $2I$ or twice as great as its normal value, and the voltage across C_y falls to zero. Current and voltage at this instant are plotted in Fig. 14-4*b*. A reflected wave reduces the voltage from E to zero, and increases current from I to $2I$, as the wave travels from right to left.

The short circuit allows unlimited current at the end of the line, but it does not allow any voltage; the boundary condition is

$$v_r = 0 \tag{14-22}$$

Hence, from Eqs. 14-7 and 14-22,

$$v_2(D + vt) = -v_1(D - vt) \tag{14-23}$$

This means that at the terminal point the reflected voltage wave will be at all times equal and opposite to the incident voltage wave. It will travel from the terminal point toward the sending end of the line as a wave with the same shape as the incident wave. The negative reflected voltage wave will be accompanied by a positive reflected current wave, in accordance with Eq. 14-19.

The total result of reflection at the short-circuited end of the line will be elimination of voltage at the end, and doubling of current. This effect is shown for a flat-topped wave in Fig. 14-4*b*, and for a rounded wave in Fig. 14-4*c*. This type of reflection is, in a sense, the inverse of that which occurs at the open end of a line, and the voltage waves in one case are exactly like the current waves in the other.

7. Reflection from Terminal Resistance. If the transmission line is neither open nor short-circuited, but has resistance connected between the receiving-end terminals, the boundary condition at the receiving end will determine the magnitude of the reflected wave. The voltage across the terminating resistance is v_r and the current through the resistance is i_r; so if R is the amount of resistance it must always be true that

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[2] It is convenient, if inelegant, to say that the wave turns about and crawls over its own back.

$$\frac{v_r}{i_r} = R \tag{14-24}$$

The amount of the terminating resistance is the factor that determines, physically, the size of the reflected wave, and must therefore do so mathematically. We know that current and voltage are related in the incident wave by

$$v_1 = Z_0 i_1 \tag{14-25}$$

and in the reflected wave by

$$v_2 = -Z_0 i_2 \tag{14-26}$$

We know further that the total voltage and current are the sums of the incident and reflected waves:

$$v = v_1 + v_2 \tag{14-27}$$

and

$$i = i_1 + i_2 \tag{14-28}$$

At the receiving end of the line, these total values of voltage and current are v_r and i_r of Eq. 14-24, and by substituting in that equation we obtain

$$\frac{v_1 + v_2}{i_1 + i_2} = R \tag{14-29}$$

Current is eliminated from this expression by means of Eqs. 14-25 and 14-26:

$$\frac{v_1 + v_2}{\dfrac{v_1}{Z_0} - \dfrac{v_2}{Z_0}} = R \tag{14-30}$$

and this equation can be solved for the reflected voltage as a function of the incident voltage:

$$v_2 = v_1 \frac{R - Z_0}{R + Z_0} \tag{14-31}$$

Equation 14-31 relates the voltage of the two waves, at the receiving end, at each instant of time. Since it is true at every instant, it is true of the functions as a whole, and it tells us that, in general, the reflected wave from a terminal resistance will be a function of time of the same form as the incident wave, but smaller by the factor of proportionality of Eq. 14-31.

It tells us, further, that if the terminating resistance is greater than the characteristic impedance Z_0, the reflected voltage will be of the same sign as the incident voltage. The limiting case is an open-ended line, for which $R = \infty$, making $v_2 = v_1$. If, on the other hand, the terminating resistance is less than the characteristic impedance, the reflected voltage wave will be negative, subtracting from the incident wave. A short circuit is the extreme case, with $R = 0$, which makes $v_2 = -v_1$.

The wave of voltage reflected from a resistance load will of course be accompanied by a reflected wave of current. Since the reflected current is related to the reflected voltage by Eq. 14-26 it will be related to the incident current by

$$i_2 = -i_1 \frac{R - Z_0}{R + Z_0} \tag{14-32}$$

8. Reflection from Terminal Inductance. If a transmission line is terminated at the receiving end with a circuit or network other than pure resistance, the differential equation of the terminating equipment must be taken into account in computing the reflected wave on the line. Consider, as a first simple example, a line with pure inductance across the receiving terminals.

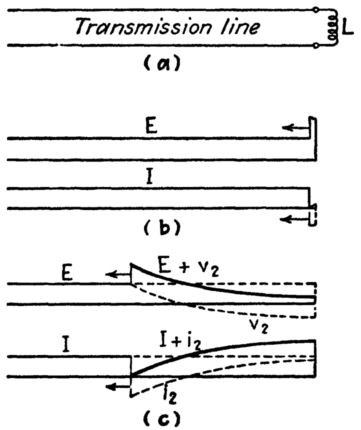

FIG. 14-5. Reflection at terminal inductance.

Figure 14-5*a* shows such a line. A traveling wave arrives at the end and is reflected. It is reflected exactly as if the line were open at the receiving end, for at the very first instant the inductance can carry no current. The terminal voltage is therefore twice the wave voltage, momentarily, and the terminal current is zero for an instant. This condition is shown in Fig. 14-5*b*.

But there is voltage across the terminal inductance from the instant of arrival of the wave, so the inductance will begin to carry current. It will carry current more and more freely, as time passes, until at last it is practically equivalent to a short circuit and carries steady current with no terminal voltage. When the voltage across the inductance has become zero, the current through it has become twice the current of the incident wave, as in the case of the short-circuited line. This condition and the transition period are shown in Fig. 14-5*c*.

As the inductance changes from an element that carries no current (and so reflects voltage with a positive sign) to an element that supports no voltage (and so reflects voltage with a negative sign), it passes through all intermediate stages and the reflected voltage changes gradually. As the reflected voltage changes, the terminal voltage across the inductance is reduced, and the rate of change of voltage is reduced also. The short-circuit condition at the terminals is therefore approached asymptotically, but never quite reached; it is reasonable to suppose that the voltage decreases according to an exponential law, and such is indeed the case.

The mathematical determination of the wave reflected from terminal inductance is quite similar to the determination when the line is ended in pure resistance. And in the more general case, with inductance and resistance in series across the end of the line, the method is still similar. In every case the reflected wave is found by simultaneous solution of the transmission-line equations 14-25 to 14-28 with the equation that expresses the boundary, or terminal, conditions. When the line is terminated in resistance, the boundary condition is expressed by Eq. 14-24; that is,

$$v_r = Ri_r \tag{14-33}$$

When there is inductance in series with the resistance, the boundary condition becomes

$$v_r = Ri_r + L\frac{di_r}{dt} \tag{14-34}$$

Operating characteristics of the transmission line are now brought into the computation by substituting in Eq. 14-34 the group of Eqs. 14-25 to 14-28; this gives the following equation which applies at the receiving end of the line:

$$v_1 + v_2 = R\left(\frac{v_1}{Z_0} - \frac{v_2}{Z_0}\right) + L\frac{d}{dt}\left(\frac{v_1}{Z_0} - \frac{v_2}{Z_0}\right) \tag{14-35}$$

Solution for v_2 in terms of v_1 gives

$$(R + Z_0)v_2 + L\frac{dv_2}{dt} = (R - Z_0)v_1 + L\frac{dv_1}{dt} \tag{14-36}$$

In this differential equation, v_1 as a function of time is presumably known, for v_1 is the incident wave that is arriving from the sending end of the line. The right-hand member of Eq. 14-36 may therefore be treated as a known function of time, and the equation is a first-order linear differential equation with constant coefficients. Its general solution is treated in calculus books or may be found in "Transient Electric Currents," Eq. 89, Chap. III (Ref. 30).

To study a simple case, let us consider that the incident wave is a flat-topped wave of constant voltage E, as in Fig. 14-5c. Equation 14-36 becomes, for such a wave,

$$(R + Z_0)v_2 + L\frac{dv_2}{dt} = (R - Z_0)E \tag{14-37}$$

This equation can be solved by separation of the variables:

$$\frac{L\,dv_2}{(R - Z_0)E - (R + Z_0)v_2} = dt \tag{14-38}$$

$$\frac{L}{-(R + Z_0)} \ln\,[(R - Z_0)E - (R + Z_0)v_2] = t + C \tag{14-39}$$

C is a constant of integration in this equation. By changing from the logarithmic form to the exponential form, and at the same time writing as K the exponential factor that contains the constant of integration, Eq. 14-39 becomes

$$(R - Z_0)E - (R + Z_0)v_2 = Ke^{-\frac{R+Z_0}{L}t} \tag{14-40}$$

The coefficient K must next be evaluated by means of known boundary conditions that relate v_2 and t. Characteristics of the terminating circuit of inductance and resistance must be considered. At the instant of arrival of the oncoming wave, the terminal circuit cannot carry finite current because of its inductance; consequently, the wave is fully reflected, and the voltage of the reflected wave (at that instant) is equal to the voltage of the incident wave. Symbolically, when

$$t = 0 \qquad i = 0$$

and therefore (14-41)

$$v_2 = E$$

It should be particularly noted that in this discussion we have arbitrarily chosen the reference time, $t = 0$, as the time of arrival of the incident wave at the receiving end of the line.

Having introduced Eq. 14-41 into 14-40, the latter becomes

$$-2Z_0E = K \tag{14-42}$$

This gives the value of K for Eq. 14-40, and finally,

$$v_2 = E\,\frac{R - Z_0 + 2Z_0e^{-\frac{R+Z_0}{L}t}}{R + Z_0} \tag{14-43}$$

Equation 14-43 does not describe a traveling wave. It gives voltage, as a function of time, at the receiving end of the line. But this voltage generates a traveling wave, and Eq. 14-43 becomes the equation of the traveling wave if $(t + \sqrt{lc}x)$ is substituted for t.

In general, Eq. 14-43 describes a wave with initial height E, for this value appears when $t = 0$. As in Fig. 14-5c, the reflected wave changes exponentially until, at very large values of time, it approaches a size determined by the load resistance alone. The effect of load inductance disappears after sufficient time.

If there were no resistance at the receiving end, but only inductance, the result would be found from Eq. 14-43 by letting $R = 0$. The reflected wave would vary from E, initially, to $-E$, finally, as in Fig. 14-5*c*.

It is interesting to notice that the reflected waves from all terminations that have been discussed (open, short-circuit, resistance, and inductance) can be found as special cases of Eq. 14-43.

9. Reflection from Terminal Capacitance. If there is a condenser connected to the terminals of the line, it acts as a short circuit at the moment of arrival of the incident wave. At this instant, the reflected voltage wave is negative, for the terminal voltage is momentarily zero, and the current of the incident wave is, momentarily, doubled. As the condenser becomes charged, the terminal voltage rises. Finally the condenser becomes fully charged; it then behaves as an open circuit, for the terminal current is zero and the voltage of the incident wave is doubled. The curves of Fig. 14-5*b* and *c* illustrate this case if the "current" curves are taken to illustrate voltage and the "voltage" waves are taken to illustrate current.

The mathematical solution for the capacitive termination is like the solution for the inductive case, and will not be given. It introduces no new ideas.

There is more practical importance in the case of a line with terminal condenser than might appear at first sight, owing to the paradox that for lightning waves a transformer acts as a capacitance rather than as an inductance. Lightning voltage arrives at a transformer as a traveling wave, and the front of the wave is so steep, and rises so suddenly to a maximum, that there is practically no time for current to start to flow through the large inductance of the transformer winding. But there is some small capacitance in the transformer from turn to turn, and from winding to core, and it is largely this capacitance, rather than the inductance, that determines the transformer's reaction to lightning.

10. Transmission-line Junctions. There must be some reflection of energy whenever any irregularity interferes with the propagation of a wave. The end of the line is an inevitable irregularity, but many others may be encountered also. Where the dimensions of a transmission line are changed, for instance, or two dissimilar lines are joined, there will be reflections as waves pass the junction.

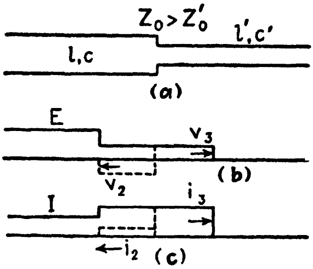

FIG. 14-6. Reflection at a change of Z_0.

Figure 14-6*a* represents the junction of a line with wide spacing and a line with narrow spacing; the former has parameters l and c, and the latter l' and c'. Since these are different, their ratios Z_0 and Z_0' will also be different. Let us assume that a wave of energy comes from the left; it enters, that is, on the widely spaced line. It has a certain ratio of voltage to current, equal to Z_0. It cannot pass the junction smoothly, for a wave on the narrowly spaced line must have a different ratio of voltage to current. So when the incident wave arrives at the junction there are two new waves formed; part of the incident energy is propagated on toward the right, and part is reflected back toward the left.

Solution for the magnitudes of the transmitted and reflected waves is possible in either of two ways. Voltage and current of the waves on one side of the junction point can be equated to voltage and current of the wave on the other side, and since ratios of voltage to current are known from the characteristic impedances it is not difficult to solve for the unknown currents and voltages. But use may be made of a previous solution by considering that, as far as the incident wave is concerned, the second section of line could be replaced by a resistance equal to Z_0'. This is possible because the current that enters the second line will be proportional at every instant to the voltage at the junction, just as it would be in a resistance. So the magnitude of the reflected voltage wave can be found from Eq. 14-31, which was derived for a terminal resistance, with Z_0' replacing R. Reflected current can be found in the same way from Eq. 14-32. Transmitted voltage is then equal to the sum of the incident and reflected waves at the junction point, and transmitted current to the sum of the incident and reflected current waves. An example is shown in *b* and *c* of Fig. 14-6.

When a line divides, as in Fig. 14-7*a*, the same methods of reasoning can be applied. The two parts of the divided line will affect the incident wave as if they were two resistances in parallel at the junction, thereby determining the reflected wave. There will be two transmitted waves, with the same voltage, but different currents as required by the different characteristic impedances. An example appears in Fig. 14-7*b* and *c*.

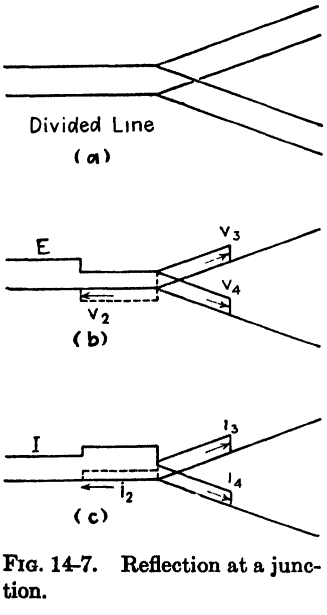

FIG. 14-7. Reflection at a junction.

Other combinations of apparatus can be handled in the same manner. Always, the requirements of all lines and circuits must be satisfied simultaneously.

11. Repeated Reflections. Only one reflection of an incident wave has been considered in the discussion thus far, but that initial reflection is in reality only the first of an infinite series. The incident wave is reflected on arrival at the receiving end of the line, and the reflected wave is reflected again at the sending end. Like a ray of light between two mirrors, the electric wave goes repeatedly from end to end of the line until its energy is exhausted.

It will be well to trace one or two typical cases.[3] Let us first consider a transmission line, open at the receiving end, to which a constant battery voltage E is suddenly applied at the sending end. The line is shown in Fig. 14-8*a*, and the incident wave in Fig. 14-8*b*. At the receiving end, the current must always be zero, so when the incident wave of current reaches the end of the line it must necessarily produce a reflected wave with negative current. The wave of negative current, when traveling toward the left, must be accompanied by a wave of positive voltage. Distribution of voltage and current shortly after reflection are shown in Fig. 14-8*c*; the solid line shows total voltage and current, and the dash line indicates the traveling wave.

When the first reflected wave arrives at the sending end of the line, it produces an unstable condition: it charges the line, as it travels, to voltage $2E$, while the sending end of the line is connected to the battery and can have no voltage other than the battery voltage E. On arrival of the wave at the sending end, the line, charged to $2E$, immediately starts to discharge again into the battery, and a traveling wave of voltage $-E$ is sent out onto the line. It must be accompanied by a current wave, and since it moves with positive velocity the current will be of the same sign as the voltage, or $-I$. Conditions shortly after reflection at the sending end are shown in Fig. 14-8*d*. For the first time, current on the line has become negative, indicating that charge is now flowing toward the battery instead of from it.

This second incident wave soon reaches the receiving end, and is again reflected. The second incident wave is a negative wave, producing negative current as it passes. But, again, there can be no current at the receiving end; so on the arrival of the second incident wave with negative current a second reflected wave with positive current must start toward the sending end. The positive current of the second reflected wave must be accompanied by a negative voltage wave (because the direction of travel is negative), and Fig. 14-8*e* shows the components of the second reflected wave. As the second reflected wave travels, it makes both total voltage and total current zero all along the line.

Just as the first reflected wave produced conditions at the sending end of the line incompatible with the boundary condition of the connected battery, so the second reflected wave leaves the line in a condition that cannot continue to exist. It leaves the sending-end voltage zero, but the sending-end voltage must be E, so an incident wave with positive voltage is sent along the line. This is the third incident wave.

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[3] Traveling waves on lines without loss are thoroughly treated by L. V. Bewley, "Traveling Waves on Transmission Systems" (Ref. 2).

The behavior of the wave need be traced no further, for the third incident wave is identical with the first incident wave; a cycle of operation has been completed, and Fig. 14-8*b* can illustrate the third incident wave, or the fifth incident wave, or the seventh, and so on. If it were true, as has been assumed, that the wave could travel without loss of energy, it would never cease to surge up and down the line.

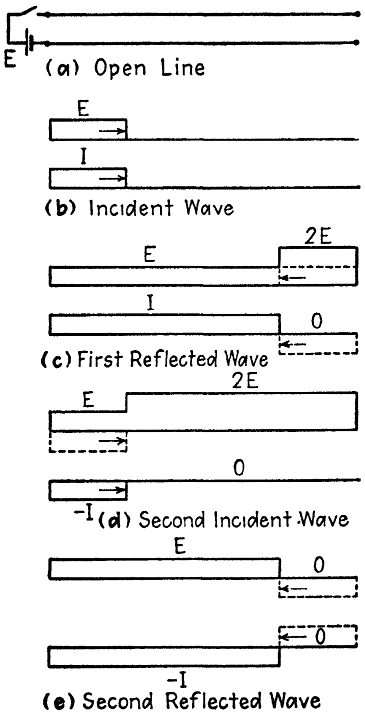

FIG. 14-8. Repeated reflections, open circuit.

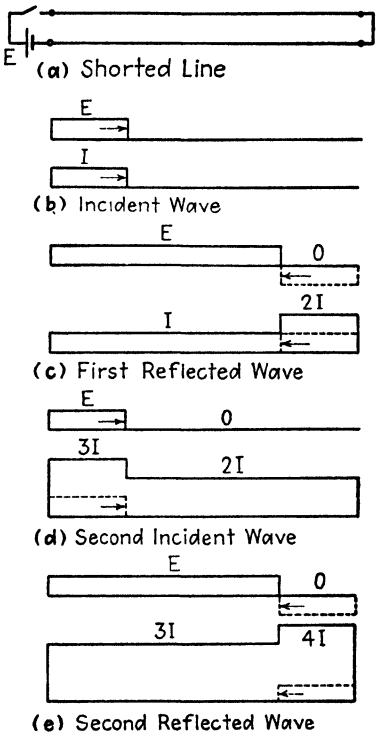

FIG. 14-9. Repeated reflections, short circuit.

A line short-circuited at the receiving end reflects waves in a different manner. Its action, neglecting energy loss, is shown in Fig. 14-9. Reflected waves are made necessary by the following boundary conditions: sending end voltage is always E and receiving end voltage is always zero.

The first reflected wave must have negative voltage and will therefore have positive current. On reaching the sending end it makes necessary a second incident wave of positive voltage, accompanied by a wave of positive current. The voltage of the line is cyclic in this simple manner: a positive wave goes out from the battery, and a negative wave returns. The current, however, is not cyclic at all, but continues to increase without limit as each reflection adds to its magnitude.

If there is resistance in the short-circuited transmission line the current will increase, but not without limit. It will approach a final value of E/rD, where r is resistance per unit length and D the length of the line. The quantity rD is merely the d-c resistance of the line, and so it obviously limits the current in the steady state. When a wave travels along a line with resistance, it becomes smaller as it travels. If it is a wave produced by the sudden application of battery voltage, as in Figs. 14-8 and 14-9, the front of the wave decreases with time according to an exponential law; the wave is therefore not flat-topped, but sloping, and slightly curved, as in Fig. 14-10*b*. Reflections of such a wave must satisfy the same terminal conditions as before: at the short-circuited receiving end, $v = 0$; at the sending end, $v = E$. A general indication of the shape of the waves and the effect of multiple reflections is given in Fig. 14-10.

It will be observed that Fig. 14-10 differs from Fig. 14-9 (which was drawn for the nondissipative line) primarily in that the traveling wave diminishes, and after several traverses of the line it becomes almost negligible. Each traverse of the line, moreover, brings the current and voltage distribution closer to the steady-state conditions, as shown in Fig. 14-10*e*.

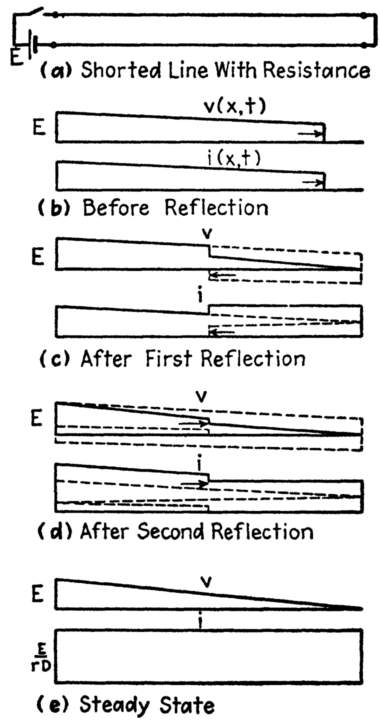
E
(a) Shorted Line With Resistance
v(x,t)
E
i(x,t)
(b) Before Reflection
v
E
i
(c) After First Reflection
v
E
i
(d) After Second Reflection
E
v
i
E/rD
(e) Steady State

FIG. 14-10. Reflection on a dissipative line.

The manner in which the wave changes as it travels is, in general, quite complicated. But there is one special case—the so-called *distortionless line*—in which the attenuation of the traveling wave is simple.

12. The Distortionless Line. If r, l, c, and g—the resistance, inductance, capacitance, and conductance per unit length, respectively—are so proportioned that

$$\frac{r}{l} = \frac{g}{c} \qquad (14\text{-}44)$$

a special case of wave transmission results. On such a line, a traveling wave from a battery will proceed outward, as in Fig. 14-10*b*, diminishing exponentially, and the voltage at a point will rise to a specific value and will remain at that value thereafter—or until the reflected wave returns. This would seem to be the natural behavior of a line when our study of the nondissipative line is considered, but it must be remembered that the nondissipative line is itself a special case that satisfies Eq. 14-44. In general the contrary is true: the voltage and current at a point will rise suddenly as the head of a wave passes and will continue to change gradually thereafter as functions of time.

When a wave travels out upon a distortionless line from a battery of voltage E, the voltage at any point on the line that has been reached by the wave, traveling with speed s, is

$$v = Ee^{-\frac{r}{l}\frac{x}{s}} \tag{14-45}$$

It will be seen that this voltage is a function of x, but not of time, indicating that after the arrival of the wave the voltage does not change. If battery voltage is applied to the sending end of a line for 0.001 second and is then removed, a wave is started out along the line. The traveling wave is shown in Fig. 14-11*a* after the front of it has progressed some 250 miles. In Fig. 14-11*b*, the applied voltage at the sending end, as a function of time, is plotted as the rectangle marked $x = 0$. If the line is distortionless, so that Eq. 14-45 applies, the voltage at a point 100 miles along the line will also be rectangular, when plotted with respect to time, but less in magnitude. It appears in the figure as the curve $x = 100$. The same may be said of the voltage where x equals 200, or 300, or at any other point. At points more distant along the line the voltage as a function of time is smaller, *but it is the same function.* This statement is not limited to waves of battery voltage, but may be applied to any sort of wave whatever; it is because of this distinctive property that the line is called *distortionless.*

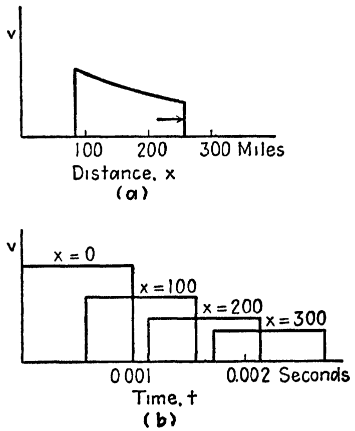

FIG. 14-11. Propagation on a distortionless line.

Another aspect of the absence of distortion is seen in the fact that the traveling wave itself (as a function of x) remains the same shape as it travels. It is important to notice that a flat-topped wave as a function of time (in Fig. 14-11*b*) is not produced by a flat-topped wave as a function of distance, but by a wave whose top is an exponential curve (Fig. 14-11*a*). As this exponential wave travels, it retains the same shape but becomes uniformly smaller with time. This, also, is true of all traveling waves on distortionless lines.

An analytical expression for voltage along a line after connection of a battery to the sending end was given in Eq. 14-45. It is of course accompanied by a current wave.

$$i = \frac{E}{Z_0} e^{-\frac{r}{l}\frac{x}{v}} \tag{14-46}$$

Proof of the correctness of Eqs. 14-45 and 14-46 is found on introducing them into the fundamental transmission-line equations 14-1 and 14-2.

13. The General Transmission Line. If the transmission line is not distortionless, its transient behavior is quite intricate. Sometimes the distortion can be neglected. In Fig. 14-10, curves were drawn for a line with resistance but no leakage—a line that is consequently not without distortion. They were drawn as if the voltage at each point of the line changed only as the head of the wave passed that point—an assumption that is contrary to fact except on a distortionless line. But in many cases the distortion is so small that the assumption does not lead to a large degree of error, and this is most likely to be true if r and g are small compared to l and c.

When there is distortion of a traveling wave, its shape, as a function of distance, is changed as it travels. A rectangular wave does not remain rectangular; a wave with exponential top, as in Fig. 14-11*a*, does not continue in quite the same form. The effect of distortion is to take energy from the head of the wave and spread it back along the line, adding to the rear part of the wave and giving it a "tail." Figure 14-12 shows how the wave of Fig. 14-11, sent out from a battery on a line that is not distortionless, will leave energy behind as it travels.

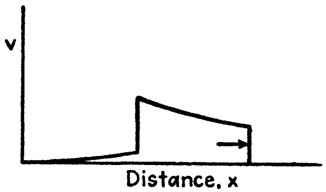

FIG. 14-12. Distortion of a traveling wave.

The main wave loses energy, on a line with distortion, not only in the resistance and conductance of the line, but also because energy is sent back along the line. As spray is driven from the front of an ocean wave by an opposing wind, to fall behind the wave or be blown back across the water, so energy is driven back from an electric wave on a line with distortion. It is only the very front of the wave that is diminished in a simple exponential manner as it travels, for it is only the front that cannot gain energy at the expense of a preceding portion of the wave. The front of a wave will always (according to the differential equations of the transmission line) travel with exactly the velocity $1/\sqrt{lc}$, regardless of loss; it will be diminished in a simple exponential manner with time, and the voltage and current at the wave front will always be related by the ratio $Z_0 = \sqrt{l/c}$.

When the line is not distortionless, each point of the line, as the wave passes, reflects continuously a small amount of energy—*every* point of the line that the wave had reached is contributing to the reflection—and the effect is distortion of the main wave and the growth of a "tail."

Derivation of the equations of traveling waves on transmission lines that involve distortion is too specialized a procedure to be given here, and the resulting equations, whether as integrals, infinite series, or Bessel functions, are not very useful unless their derivations are understood. Many books treat the subject quite fully and may be used as reference.[4]

Skin effect has an interesting influence on transient traveling waves. Skin effect is usually considered to be important at high frequency. More basically, it is important when the time rate of change of current is high. At a steep wave front, the rate of change of current is extremely high. The front of a traveling square wave, therefore, may encounter markedly higher resistance and slightly lower inductance than does the rest of the same wave, owing to the fact that current just at the front of the wave does not penetrate very deeply into the conductor. This variability of r and l contributes slightly to the distortion of a traveling wave, particularly in rounding the sharp corners of the wave front.[5]

14. Speed of Wave Travel. It has been stated in this chapter that traveling waves have a definite velocity, determined by the parameters of the line, equal to $1/\sqrt{lc}$. This seems, at first, to be at variance with the well-known fact that, in the steady state, sinusoidal waves of low frequency appear to travel more slowly than sinusoidal waves of high frequency. But the lower speed in the steady state is merely apparent, not real, for it is the combined result of true velocity and distortion. If there is no distortion, r, g, l, and c being properly related, the velocity of all waves is $1/\sqrt{lc}$.

When a sinusoidal wave of a steady-state wave-train is distorted as it travels, the distortion takes a very special and peculiar form. Energy is removed from the front of the wave, and added to the back of the wave, by the distortion, in such a way that the distorted wave is still sinusoidal but appears not to have progressed as far as a square wave, for example, would obviously have done.

This type of distortion, which does not appear to be distortion but does appear to be a change of velocity, can take place only in the steady state. Moreover, it can take place only in a sinusoidal wave. If the steady-state wave is not sinusoidal, it will be distorted in such a way that each sinusoidal component will appear to have a different speed. A single traveling wave will never appear to have reduced velocity, for it is not subject to this type of distortion.

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[4] For instance, V. Bush, "Operational Circuit Analysis" (Ref. 4); or L. F. Woodruff, "Principles of Electric Power Transmission" (Ref. 38).

[5] Another type of distortion not to be derived from the differential equations of the transmission line is treated by H. H. Skilling and P. de K. Dykes, Distortion of Traveling Waves by Corona, *Trans. AIEE*, 1937, p. 850.

15. Transition to Steady State. The traveling waves that run back and forth many times on a transmission line after the sudden application of voltage (or other sudden change) constitute a transient state of operation that merges gradually into the steady state. Every reflection of the transient traveling wave, since it must necessarily be such as to satisfy the terminal boundary conditions, brings the total distribution of voltage and current on the line closer to the steady-state distribution. In Fig. 14-10, for example, each transient condition shown brings voltage and current distribution closer to the steady state, and the traveling waves grow continually smaller. Eventually the traveling waves disappear, and the steady state is established.

In Fig. 14-10, a battery voltage is applied and the steady state is direct current. If the applied voltage were alternating, the situation would be only a little more complicated.

Assume a line that is several wavelengths long, with small but appreciable losses, and terminal equipment at each end with some arbitrary amount of impedance. Alternating voltage is suddenly applied at the sending end; a traveling wave goes out onto the line. It has an abruptly rising front the height of which is equal to the instantaneous voltage at the instant of application, and thereafter a sinusoidal form. When the wave front reaches the receiving end of the line, there is partial reflection and a smaller wave returns. When the front of the reflected wave reaches the sending end, there is again partial reflection, and a still smaller wave starts toward the load. When this arrives at the receiving end, there is again partial reflection, and so on. Each wave grows smaller as it travels, and each reflection reduces the size of the wave.

Each of the many traveling waves adds a small alternating component to the total voltage and current at every point on the line. At every point, therefore, the voltage and current approach their steady-state values by an infinite series of steps. (Only in a completely nondissipative system will the traveling waves go on forever.)

Eventually, then, the steady state is reached. In the alternating steady state, as considered in Chap. 1, the net effect is one steady sinusoidal wave traveling from generator to load, and one reflected wave from load to generator. The steady-state incident wave is the sum of all the transient wavelets traveling from generator toward load—the initial transient wave and all the later partial reflections from the generator end. The steady-state reflected wave is similarly made up of all the series of transient wavelets reflected from the load.

Mathematically, consider that a sinusoidal voltage V_s is suddenly applied to the sending end of a line (dissipative but distortionless) and a traveling wave starts out upon the line. The reflection factor at the receiving end is ρ, and ρ may have any complex value. Let us assume that the generator is without impedance, so that any wave arriving at the transmitting end of the line is totally reflected with reversal of voltage; the reflection factor at the sending end is thus -1. The traveling wave will follow a schedule such as this:

Starts from generator: V_s
 Arrives at receiver: $V_s e^{-jBx}$
 Reflected from receiver: $\rho V_s e^{-jBx}$
Arrives at generator: $\rho V_s e^{-j2Bx}$
Reflected from generator: $-\rho V_s e^{-j2Bx}$
 Arrives at receiver: $-\rho V_s e^{-j3Bx}$
 Reflected from receiver: $-\rho^2 V_s e^{-j3Bx}$
Arrives at generator: $-\rho^2 V_s e^{-j4Bx}$
Reflected from generator: $\rho^2 V_s e^{-j4Bx}$
 Arrives at receiver: $\rho^2 V_s e^{-j5Bx}$
 Etc.

The total voltage of the steady-state incident wave at the receiver is found by adding the voltages of the many traveling wavelets that arrive at the receiving end:

$$\begin{aligned} V_1 &= V_s e^{-jBx} - \rho V_s e^{-j3Bx} + \rho^2 V_s e^{-j5Bx} - \cdots \\ &= V_s e^{-jBx}(1 - \rho e^{-j2Bx} + \rho^2 e^{-j4Bx} - \cdots) \end{aligned} \tag{14-47}$$

The steady state is found by allowing time for an unlimited number of reflections. The series in parentheses is then an infinite series, and this infinite series is recognized as the expansion of a simple function, making

$$V_1 = V_s e^{-jBx}\left(\frac{1}{1 + \rho e^{-j2Bx}}\right) \tag{14-48}$$

from which

$$V_s = V_1(e^{jBx} + \rho e^{-jBx}) = V_1 e^{jBx} + V_2 e^{-jBx} \tag{14-49}$$

Thus V_1, the sum of all the transient wavelets that arrive at the receiving end of the line, is identified with the steady-state incident wave, and similarly the sum of all the transient wavelets that travel toward the generator can be identified with V_2.

If the generator has internal impedance, the behavior of the traveling waves is much the same. An appropriate reflection factor then appears when transient waves are reflected at the generator end of the line and, as in the above example, the sum of the transient waves converges on the steady state.

Thus nature is ingeniously able to satisfy conditions that must exist at the generator, and at the same time to satisfy other conditions that must be met at the load, and all the while to satisfy the transmission-line requirements symbolized by the partial differential equations. Such is the response of the line to any arbitrary disturbance that one may choose to impose. The beauty of this phenomenon is worth contemplation.

PROBLEMS

14-1. A transmission line is uniformly charged, from end to end, to voltage E. Both ends are open. One end is then suddenly short-circuited. Plot current at the short-circuited end as a function of time. Plot both voltage and current at the mid-point of the line as functions of time. Neglect loss.

14-2. A transmission line (uncharged) is open-circuited at both ends. A battery of constant voltage E is suddenly connected across the line at a point one-third of the length from one end. Plot battery current as a function of time. Plot voltage at one end of the line as a function of time. Neglect loss.

14-3. A transmission line is terminated in resistance equal to $2Z_0$. The internal impedance of the battery (Fig. 14-8) is also $2Z_0$. Plot current and voltage along the line shortly after the first, second, third, and fourth reflections.

14-4. A transmission line is terminated in resistance equal to $\frac{1}{2}Z_0$. The internal impedance of the battery (Fig. 14-8) is $2Z_0$. Plot current and voltage along the line shortly after the first, second, third, and fourth reflections.

14-5. A transmission line has neither charge nor current. A battery of constant voltage E is suddenly connected into one of the line conductors at the mid-point of the line (a series, not a shunt, connection). Sketch the waves that start along the line (shortly after the battery is inserted), and show that they satisfy Eqs. 14-18 and 14-19 as well as give continuity of voltage and current.

14-6. Plot battery current in Fig. 14-10 as a function of time. Do the same for an open-circuited line with moderate losses.

14-7. Repeat Prob. 14-1, giving approximate curves for current and voltage if there is a moderate amount of line loss. Show how the steady state is approached.

14-8. Repeat Prob. 14-2, giving approximate curves for current and voltage if there is a moderate amount of line loss. Show how the steady state is approached.

14-9. A transmission line is terminated in a series-resonant circuit of inductance and capacitance with a natural period much less than the time required for a wave to travel the length of the line. Sketch current and voltage along the line at several instants after reflection. A qualitative solution is sufficient.

14-10. Simulating the result of lightning, charge is placed on one section of a transmission line. That section (isolated by a disconnecting switch at each end) is uniformly charged to voltage E, and the rest of the line is uncharged. Charge is suddenly allowed to flow from the charged section to the rest of the line (by simultaneously closing the disconnecting switches). Plot voltage and current at several later instants to show the waves that travel in both directions.

14-11. An isolated line is uniformly charged to voltage E. A resistance equal to Z_0 is suddenly connected across one end of the line. Plot voltage across the resistance as a function of time.

14-12. A pulse-forming circuit (as in radar) uses the principle of the line of Prob. 14-11, but employs an artificial line rather than a line with distributed constants. The circuit is similar to that of Fig. 14-1, with a resistance equal to Z_0 substituted for the battery. Design a line, with $Z_0 = 50$ ohms, to give a 1-microsecond pulse; use four sections of artificial line.

14-13. Considering the artificial line of Prob. 14-12 as a filter, what is its cutoff frequency? How does this affect the shape of the pulses obtained?

14-14. High-voltage traveling waves, notably those caused by lightning, are rapidly reduced in size as energy is dissipated in corona loss. Discuss the differences between this effect and the type of loss included in Eqs. 14-1 and 14-2 [see Sec. 13, Chap. 9; also H. H. Skilling and P. de K. Dykes, Distortion of Traveling Waves by Corona, *Trans. AIEE*, 1937, p. 850, or Westinghouse "Reference Book," p. 287 (Ref. 18)].

APPENDIX 1

SINES, COSINES, AND TANGENTS OF NUMBERS

This is a condensed table, making use of the repetitive nature of the trigonometric functions. The three right-hand columns give sin x, cos x, and tan x; the first column on the left gives x if the value of x is between 0 and $\pi/2$. Since the trigonometric functions repeat, except for change of sign, as x increases from $\pi/2$ to π, from π to $3\pi/2$, and again from $3\pi/2$ to 2π, the values of the argument x for these various ranges can be tabulated in the second, third, and fourth columns from the left. The table is then useful for any value of x from 0 to 2π. However, the values of x in the second, third, and fourth columns are not rational numbers; they are given to five significant figures (the last two significant figures, being the same throughout each column, are omitted from four out of each group of five numbers) and a good degree of accuracy is possible by careful interpolation. If only three significant figures are used, and interpolation is not done, the accuracy is nearly that of ordinary slide-rule computation, and this is often as good as physical data warrant.

The algebraic sign of the trigonometric functions alternates periodically, and the sign to be used is indicated at the head of each column: thus, $t+$ indicates that the tangent is positive for values of x in the column below.

Since the functions are repetitive in both sign and magnitude, any multiple of 2π may be subtracted from or added to any value of x, and by this means any argument may be brought within the range of this table.

Summary of Instructions for Use of Table

If x is between 0 and 1.5708, read x in column 1.
If between 1.5708 and 3.1416, read x in column 2.
If between 3.1416 and 4.7124, read x in column 3.
If between 4.7124 and 6.2832, read x in column 4.
For higher values of x, subtract even multiples of 6.2832.

The sign to be attached to *sin* x, *cos* x, and *tan* x is indicated at the head of the appropriate column for x. If x is negative, use the table as if for positive x but reverse the sign of sine and tangent.

Example:

$$\tan 2.1516 = -1.5237$$

$s+$ $c+$ $t+$	$s+$ $c-$ $t-$	$s-$ $c-$ $t+$	$s-$ $c+$ $t-$	sin x	cos x	tan x
0.00	3.1416	3.1416	6.2832	0.0000	1.0000	0.0000
01	13	15	27	0100	1.0000	0100
02	12	16	26	0200	0.9998	0200
03	11	17	25	0300	9996	0300
04	10	18	24	0400	9992	0400
0.05	3.0916	3.1916	6.2332	0.0500	0.9988	0.0500
06	08	20	22	0600	9982	0601
07	07	21	21	0699	9976	0701
08	06	22	20	0799	9968	0802
09	05	23	19	0899	9960	0902
0.10	3.0416	3.2416	6.1832	0.0998	0.9950	0.1003
11	03	25	17	1098	9940	1105
12	02	26	16	1197	9928	1206
13	01	27	15	1296	9916	1307
14	3.0016	28	14	1395	9902	1409
0.15	2.9916	3.2916	6.1332	0.1494	0.9888	0.1511
16	98	30	12	1593	9872	1614
17	97	31	11	1692	9856	1717
18	96	32	10	1790	9838	1820
19	95	33	09	1889	9820	1923
0.20	2.9416	3.3416	6.0832	0.1987	0.9801	0.2027
21	93	35	07	2085	9780	2131
22	92	36	06	2182	9759	2236
23	91	37	05	2280	9737	2341
24	90	38	04	2377	9713	2447
0.25	2.8916	3.3916	6.0332	0.2474	0.9689	0.2553
26	88	40	02	2571	9664	2660
27	87	41	01	2667	9638	2768
28	86	42	6.0032	2764	9611	2876
29	85	43	5.9932	2860	9582	2984
0.30	2.8416	3.4416	5.9832	0.2955	0.9553	0.3093
31	83	45	97	3051	9523	3203
32	82	46	96	3146	9492	3314
33	81	47	95	3240	9460	3425
34	80	48	94	3335	9428	3537
0.35	2.7916	3.4916	5.9332	0.3429	0.9394	0.3650
36	78	50	92	3523	9359	3764
37	77	51	91	3616	9323	3879
38	76	52	90	3709	9287	3994
39	75	53	89	3802	9249	4111

x						
$s+$ $c+$ $t+$	$s+$ $c-$ $t-$	$s-$ $c-$ $t+$	$s-$ $c+$ $t-$	sin x	cos x	tan x
0.40	2.7416	3.5416	5.8832	0.3894	0.9211	0.4228
41	73	55	87	3986	9171	4346
42	72	56	86	4078	9131	4466
43	71	57	85	4169	9090	4586
44	70	58	84	4259	9048	4708
0.45	2.6916	3.5916	5.8332	0.4350	0.9004	0.4831
46	68	60	82	4439	8961	4955
47	67	61	81	4529	8916	5080
48	66	62	80	4618	8870	5206
49	65	63	79	4706	8823	5334
0.50	2.6416	3.6416	5.7832	0.4794	0.8776	0.5463
51	63	65	77	4882	8727	5594
52	62	66	76	4969	8678	5726
53	61	67	75	5055	8628	5859
54	60	68	74	5141	8577	5994
0.55	2.5916	3.6916	5.7332	0.5227	0.8525	0.6131
56	58	70	72	5312	8473	6270
57	57	71	71	5396	8419	6410
58	56	72	70	5480	8365	6552
59	55	73	69	5564	8309	6696
0.60	2.5416	3.7416	5.6832	0.5646	0.8253	0.6841
61	53	75	67	5729	8196	6989
62	52	76	66	5810	8139	7139
63	51	77	65	5891	8080	7291
64	50	78	64	5972	8021	7445
0.65	2.4916	3.7916	5.6332	0.6052	0.7961	0.7602
66	48	80	62	6131	7900	7761
67	47	81	61	6210	7838	7922
68	46	82	60	6288	7776	8087
69	45	83	59	6365	7712	8253
0.70	2.4416	3.8416	5.5832	0.6442	0.7648	0.8423
71	43	85	57	6518	7584	8595
72	42	86	56	6594	7518	8771
73	41	87	55	6669	7452	8949
74	40	88	54	6743	7385	9131
0.75	2.3916	3.8916	5.5332	0.6816	0.7317	0.9316
76	38	90	52	6889	7248	9504
77	37	91	51	6961	7179	9697
78	36	92	50	7033	7109	0.9893
79	35	93	49	7104	7038	1.0092

x						
$s+$ $c+$ $t+$	$s+$ $c-$ $t-$	$s-$ $c-$ $t+$	$s-$ $c+$ $t-$	$\sin x$	$\cos x$	$\tan x$
0.80	2.3416	3.9416	5.4832	0.7174	0.6967	1.0296
81	33	95	47	7243	6895	0505
82	32	96	46	7311	6822	0717
83	31	97	45	7379	6749	0934
84	30	98	44	7446	6675	1156
0.85	2.2916	3.9916	5.4332	0.7513	0.6600	1.1383
86	28	4.0016	42	7578	6524	1616
87	27	01	41	7643	6448	1853
88	26	02	40	7707	6372	2097
89	25	03	39	7771	6294	2346
0.90	2.2416	4.0416	5.3832	0.7833	0.6216	1.2602
91	23	05	37	7895	6137	2864
92	22	06	36	7956	6058	3133
93	21	07	35	8016	5978	3409
94	20	08	34	8076	5898	3692
0.95	2.1916	4.0916	5.3332	0.8134	0.5817	1.3984
96	18	10	32	8192	5735	4284
97	17	11	31	8249	5653	4592
98	16	12	30	8305	5570	4910
0.99	15	13	29	8360	5487	5237
1.00	2.1416	4.1416	5.2832	0.8415	0.5403	1.5574
01	13	15	27	8468	5319	5922
02	12	16	26	8521	5234	6281
03	11	17	25	8573	5148	6652
04	10	18	24	8624	5062	7036
1.05	2.0916	4.1916	5.2332	0.8674	0.4976	1.7433
06	08	20	22	8724	4889	7844
07	07	21	21	8772	4801	8270
08	06	22	20	8820	4713	8712
09	05	23	19	8866	4625	9171
1.10	2.0416	4.2416	5.1832	0.8912	0.4536	1.9648
11	03	25	17	8957	4447	2.0143
12	02	26	16	9001	4357	0660
13	01	27	15	9044	4267	1198
14	2.00	28	14	9086	4176	1759
1.15	1.9916	4.2916	5.1332	0.9128	0.4085	2.2345
16	98	30	12	9168	3993	2958
17	97	31	11	9208	3902	3600
18	96	32	10	9246	3809	4273
19	95	33	09	9284	3717	4979

x						
s+ *c*+ *t*+	*s*+ *c*− *t*−	*s*− *c*− *t*+	*s*− *c*+ *t*−	sin *x*	cos *x*	tan *x*
1.20	1.9416	4.3416	5.0832	0.9320	0.3624	2.5722
21	93	35	07	9356	3530	6503
22	92	36	06	9391	3436	7328
23	91	37	05	9425	3342	8198
24	90	38	04	9458	3248	2.9119
1.25	1.8916	4.3916	5.0332	0.9490	0.3153	3.0096
26	88	40	02	9521	3058	1133
27	87	41	01	9551	2963	2236
28	86	42	5.0032	9580	2867	3413
29	85	43	4.9932	9608	2771	4672
1.30	1.8416	4.4416	4.9832	0.9636	0.2675	3.6021
31	83	45	97	9662	2578	3.7471
32	82	46	96	9687	2482	3.9033
33	81	47	95	9711	2385	4.0723
34	80	48	94	9735	2288	4.2556
1.35	1.7916	4.4916	4.9332	0.9757	0.2190	4.4552
36	78	50	92	9779	2092	4.6734
37	77	51	91	9799	1994	4.9131
38	76	52	90	9818	1896	5.1774
39	75	53	89	9837	1798	5.4707
1.40	1.7416	4.5416	4.8832	0.9854	0.1700	5.7979
41	73	55	87	9871	1601	6.1654
42	72	56	86	9887	1502	6.5811
43	71	57	85	9901	1403	7.0555
44	70	58	84	9915	1304	7.6018
1.45	1.6916	4.5916	4.8332	0.9927	0.1205	8.2381
46	68	60	82	9939	1106	8.9886
47	67	61	81	9949	1006	9.8874
48	66	62	80	9959	0907	10.983
49	65	63	79	9967	0807	12.350
1.50	1.6416	4.6416	4.7832	0.9975	0.0707	14.101
51	63	65	77	9982	0608	16.428
52	62	66	76	9987	0508	19.670
53	61	67	75	9992	0408	24.498
54	60	68	74	9995	0308	32.461
1.55	1.5916	4.6916	4.7332	0.9998	0.0208	48.078
1.56	1.5816	4.7016	4.7232	0.9999	0.0108	92.621
1.57	1.5716	4.7116	4.7132	1.0000	0.0008	1255.8
1.5708		4.7124		1.0000	0.0000	Inf.

APPENDIX 2

HYPERBOLIC AND EXPONENTIAL FUNCTIONS

x	$\sinh x$	$\cosh x$	$\tanh x$	e^x	e^{-x}
0.00	0.0000	1.0000	0.0000	1.0000	1.0000
01	0100	0001	0100	0101	0.9900
02	0200	0002	0200	0202	9802
03	0300	0004	0300	0305	9704
04	0400	0008	0400	0408	9608
0.05	0.0500	1.0012	0.0500	1.0513	0.9512
06	0600	0018	0599	0618	9418
07	0701	0024	0699	0725	9324
08	0801	0032	0798	0833	9231
09	0901	0040	0898	0942	9139
0.10	0.1002	1.0050	0.0997	1.1052	0.9048
11	1102	0061	1096	1163	8958
12	1203	0072	1194	1275	8869
13	1304	0085	1293	1388	8781
14	1405	0098	1391	1503	8694
0.15	0.1506	1.0113	0.1489	1.1618	0.8607
16	1607	0128	1587	1735	8521
17	1708	0145	1684	1853	8437
18	1810	0162	1781	1972	8353
19	1912	0181	1878	2092	8270
0.20	0.2013	1.0201	0.1974	1.2214	0.8187
21	2116	0221	2070	2337	8106
22	2218	0243	2165	2461	8025
23	2320	0266	2260	2586	7945
24	2423	0289	2355	2712	7866
0.25	0.2526	1.0314	0.2449	1.2840	0.7788
26	2629	0340	2543	2969	7711
27	2733	0367	2636	3100	7634
28	2837	0395	2729	3231	7558
29	2941	0424	2821	3364	7483
0.30	0.3045	1.0453	0.2913	1.3499	0.7408
31	3150	0484	3004	3634	7334
32	3255	0516	3095	3771	7261
33	3360	0550	3185	3910	7189
34	3466	0584	3275	4049	7118

x	$\sinh x$	$\cosh x$	$\tanh x$	e^x	e^{-x}
0.35	0.3572	1.0619	0.3364	1.4191	0.7047
36	3678	0655	3452	4333	6977
37	3785	0692	3540	4477	6907
38	3892	0731	3627	4623	6839
39	4000	0770	3714	4770	6771
0.40	0.4108	1.0811	0.3800	1.4918	0.6703
41	4216	0852	3885	5068	6637
42	4325	0895	3969	5220	6570
43	4434	0939	4053	5373	6505
44	4543	0984	4136	5527	6440
0.45	0.4653	1.1030	0.4219	1.5683	0.6376
46	4764	1077	4301	5841	6313
47	4875	1125	4382	6000	6250
48	4986	1174	4462	6161	6188
49	5098	1225	4542	6323	6126
0.50	0.5211	1.1276	0.4621	1.6487	0.6065
51	5324	1329	4700	6653	6005
52	5438	1383	4777	6820	5945
53	5552	1438	4854	6989	5886
54	5666	1494	4930	7160	5827
0.55	0.5782	1.1551	0.5005	1.7333	0.5770
56	5897	1609	5080	7507	5712
57	6014	1669	5154	7683	5655
58	6131	1730	5227	7860	5599
59	6248	1792	5299	8040	5543
0.60	0.6366	1.1855	0.5370	1.8221	0.5488
61	6485	1919	5441	8404	5434
62	6605	1984	5511	8589	5379
63	6725	2051	5580	8776	5326
64	6846	2119	5649	8965	5273
0.65	0.6968	1.2188	0.5717	1.9155	0.5220
66	7090	2258	5784	9348	5169
67	7213	2330	5850	9542	5117
68	7336	2402	5915	9739	5066
69	7461	2476	5980	9937	5016
0.70	0.7586	1.2552	0.6044	2.0138	0.4966
71	7712	2628	6107	0340	4916
72	7838	2706	6169	0544	4868
73	7966	2785	6231	0751	4819
74	8094	2865	6292	0959	4771
0.75	0.8223	1.2947	0.6352	2.1170	0.4724
76	8353	3030	6411	1383	4677
77	8484	3114	6469	1598	4630
78	8615	3199	6527	1815	4584
79	8748	3286	6584	2034	4538

x	sinh x	cosh x	tanh x	e^x	e^{-x}
0.80	0.8881	1.3374	0.6640	2.2255	0.4493
81	9015	3464	6696	2479	4449
82	9150	3555	6751	2705	4404
83	9286	3647	6805	2933	4360
84	9423	3740	6858	3164	4317
0.85	0.9561	1.3835	0.6911	2.3396	0.4274
86	9700	3932	6963	3632	4232
87	9840	4029	7014	3869	4190
88	9981	4128	7064	4109	4148
89	1.0122	4229	7114	4351	4107
0.90	1.0265	1.4331	0.7163	2.4596	0.4066
91	0409	4434	7211	4843	4025
92	0554	4539	7259	5093	3985
93	0700	4645	7306	5345	3946
94	0847	4753	7352	5600	3906
0.95	1.0995	1.4862	0.7398	2.5857	0.3867
96	1144	4973	7443	6117	3829
97	1294	5085	7487	6379	3791
98	1446	5199	7531	6645	3753
99	1598	5314	7574	6912	3716
1.00	1.1752	1.5431	0.7616	2.7183	0.3679
1.10	3356	6685	8005	3 0042	3329
1.20	5095	8107	8336	3.3201	3012
1.30	6984	9709	8617	3.6693	2725
1.40	9043	2.1509	8854	4.0552	2466
1.50	2.1293	2.3524	0.9052	4.4817	0.2231
1.60	3756	5775	9217	4.9530	2019
1.70	6456	8283	9354	5.4739	1827
1.80	9422	3.1075	9468	6.0496	1653
1.90	3.2682	4177	9562	6.6859	1496
2.00	3.6269	3.7622	0.9640	7.3891	0.1353
2.10	4.0219	4.0219	9704	8.1662	1225
2.20	4.4571	4.4571	9757	9.0250	1108
2.30	4.9370	4.9370	9801	9.9742	1003
2.40	5.4662	5.4662	9837	11.023	0907
2.50	6.0502	6.1323	0.9866	12.182	0.0821
2.60	6.6947	6.7690	9890	13.464	0743
2.70	7.4063	7.4735	9910	14.880	0672
2.80	8.1919	8.2527	9926	16.445	0608
2.90	9.0596	9.1146	9940	18.174	0550
3.00	10.0179	10.0677	0.9950	20.086	0.0498
3.50	16.5426	16.5728	9982	33.115	0302
4.00	27.2899	27.3082	9993	54.598	0183
4.50	45.0030	45.0141	9998	90.017	0111
5.00	74.2032	74.2099	9999	148.41	0067

APPENDIX 3

POWER-LINE REACTANCE TABLES

Conductor component x_a and spacing component x_d of 60-cycle reactance and conductor resistance are given in the following Tables A3-1, A3-2, A3-3, and A3-4.

Conductor component x_a' and spacing component x_d' of 60-cycle *shunt* capacitive reactance are given in Tables A3-1, A3-2, A3-5, and A3-6.

Three-phase Line. The ordinary impedance of a three-phase line is $z_1 = r_a + j(x_a + x_d)$ ohms (to neutral) per mile of conductor (see Sec. 12, Chap. 6).

The ordinary shunt capacitive reactance is $x_1' = x_a' + x_d'$, the reciprocal of which (if the line has negligible shunt conductance) is the shunt admittance y; x_1' is given in megohms to neutral per mile of conductor, yielding y in micromhos to neutral per mile of conductor (see Sec. 15, Chap. 6).

These, z_1 and x_1', are positive sequence quantities. For studies of unbalanced currents, the negative-sequence quantities z_2 and x_2', are used, and also the zero-sequence quantities z_0 and x_0' (see equations with Tables A3-3 and A3-5).

Single-phase Line. Impedance of a single-phase circuit of two identical conductors is $z = 2z_1$ ohms per mile of two-wire line. The shunt capacitive reactance is $x' = 2x_1$ megohms per mile of two-wire line.

Impedance of a single-phase circuit of one conductor and earth return is $r_a + \frac{1}{3}r_e + j(x_a + \frac{1}{3}x_e)$. The shunt capacitive reactance of such an earth-return circuit is $x_a' + \frac{1}{3}x_e'$.

The tables of this appendix are reproduced by courtesy of the Westinghouse Electric Company from the Westinghouse "Reference Book" (Ref. 18). More complete tables, and data for other types of conductors, will be found in that book. Tables showing the conductor component and the spacing component of reactance separately were first published by Wagner and Evans (Ref. 36), giving credit for the arrangement to W. A. Lewis (Ref. 16).

TABLE A3-1

CHARACTERISTICS OF COPPER CONDUCTORS, HARD DRAWN, 97.3 PER CENT CONDUCTIVITY

Size of conductor		Number of strands	Diameter of individual strands, inches	Outside diameter, inches	Approx. current-carrying capacity,* amp.	r_a, resistance ohms per conductor per mile				x_a, ohms per mile	x_a', megohms per mile
Circular mils	AWG or B & S					25°C (77°F)		50°C (122°F)			
						D-c	60 cycles	D-c	60 cycles	60 cycles	60 cycles
1 000 000		37	0 1644	1 151	1 300	0 0585	0 0634	0 0640	0 0685	0 400	0 0901
900 000		37	1560	1 092	1 220	0650	0695	0711	0752	406	0916
800 000		37	1470	1 029	1 130	0731	0772	0800	0837	413	0934
750 000		37	1424	997	1 090	0780	0818	0853	0888	417	0943
700 000		37	1375	963	1 040	0836	0871	0914	0947	422	0954
600 000		37	1273	891	940	0975	1006	1066	1095	432	0977
500 000		37	1162	814	840	1170	1196	1280	1303	443	1004
500 000		19	1622	.811	840	1170	1196	1280	1303	445	1005
450 000		19	1539	770	780	1300	1323	1422	1443	451	1020
400 000		19	1451	726	730	1462	1484	1600	1619	458	1038
350 000		19	1357	679	670	1671	1690	1828	1845	466	1058
350 000		12	1708	710	670	1671	1690	1828	1845	460	1044
300 000		19	1257	629	610	1950	1966	213	215	476	1080
300 000		12	1581	657	610	1950	1966	213	215	470	1068
250 000		19	1147	574	540	234	235	256	257	487	1108
250 000		12	1443	600	540	234	235	256	257	481	1094
211 600	4/0	19	1055	528	480	276	278	302	303	497	1132
211 600	4/0	12	1328	552	490	276	278	302	303	491	1119
211 600	4/0	7	1739	522	480	276	278	302	303	503	1136
167 800	3/0	12	1183	492	420	349	350	381	382	505	1153
167 800	3/0	7	1548	464	420	349	350	381	382	518	1171
133 100	2/0	7	1379	414	360	440	440	481	481	532	1205
105 500	1/0	7	1228	368	310	555	555	606	607	546	1240
83 690	1	7	1093	328	270	699	699	765		560	1274
83 690	1	3	1670	360	270	692	692	757		557	1246
66 370	2	7	0974	292	230	881	882	964		574	1308
66 370	2	3	1487	320	240	873		955		571	1281
66 370	2	1		258	220	864		945		581	1345
52 630	3	7	0867	260	200	1 112	Same as d-c	1 216	Same as d-c	588	1343
52 630	3	3	1325	285	200	1 101		1 204		585	1315
52 630	3	1	.	229	190	1 090		1 192		595	1380
41 740	4	3	1180	254	180	1 388		1 518		599	1349
41 740	4	1	.	204	170	1 374		1 503		609	1415
33 100	5	3	1050	226	150	1 750		1 914		613	1384
33 100	5	1		1819	140	1 733		1 895		623	1449
26 250	6	3	0935	201	130	2 21		2 41		628	1419
26 250	6	1	....	1620	120	2 18		2 39		637	1483
20 820	7	1	.	1443	110	2 75		3 01		651	1517
16 510	8	1	.. .	1285	90	3 47		3 80		665	1552

*For conductor at 75°C, air at 25°C, wind 1.4 miles per hour (2 feet per second); frequency = 60 cycles.

TABLE A3-2

CHARACTERISTICS OF ALUMINUM CABLE STEEL REINFORCED*

Circular mils or AWG aluminum	Aluminum			Steel		Outside diameter, inches	Approx. current-carrying capacity† amp.	r_a, resistance ohms per conductor per mile				x_a, ohms per mile	x_a', megohms per mile
	Strands	Layers	Strand diam., inches	Strands	Strand diam., inches			25°C (77°F)		50°C (122°F)			
								D-c	60 cycles	D-c	60 cycles	60 cycles	60 cycles
1 590 000	54	3	.1716	19	.1030	1 545	1 380	0 0587	0.0591	0 0646	0.0684	0.359	0 0814
1 510 500	54	3	.1673	19	.1004	1.506	1 340	.0618	.0622	.0680	.0720	.362	.0821
1 431 000	54	3	1628	19	0977	1 465	1 300	0652	0656	.0718	.0760	.365	.0830
1 351 000	54	3	.1582	19	0949	1.424	1 250	0691	0695	.0761	.0803	.369	.0838
1 272 000	54	3	.1535	19	0921	1 382	1 200	0734	.0738	.0808	.0851	.372	.0847
1 192 500	54	3	.1486	19	.0892	1.338	1 160	.0783	.0788	.0862	.0906	.376	.0857
1 113 000	54	3	1436	19	0862	1.293	1 110	0839	.0844	.0924	.0969	.380	.0867
1 033 500	54	3	1384	7	.1384	1 246	1 060	.0903	0909	.0994	.1035	.385	.0878
954 000	54	3	.1329	7	1329	1.196	1 010	0979	0982	.1078	.1128	.390	.0890
900 000	54	3	.1291	7	.1291	1 162	970	104	104	.1145	.1185	.393	.0898
874 500	54	3	.1273	7	1273	1 146	950	107	.108	.1178	.1228	.395	.0903
795 000	54	3	.1214	7	.1214	1.093	900	117	.119	.1288	.1378	.401	.0917
795 000	26	2	.1749	7	1360	1.108	900	.117	117	.1288	.1288	.399	.0912
795 000	30	3	.1628	19	0977	1 140	910	.117	.117	.1288	.1288	.393	.0904
715 500	54	3	.1151	7	1151	1 036	830	.131	132	.1442	.1482	.407	.0932
715 500	26	2	.1659	7	1290	1 051	840	.131	.131	.1442	1442	.405	.0928
715 500	30	2	.1544	19	0926	1 081	840	131	.131	.1442	1442	.399	.0920
666 600	54	3	.1111	7	1111	1.000	800	.140	.141	.1541	.1601	.412	.0943
636 000	54	3	.1085	7	1085	0.977	770	.147	.148	.1618	.1688	.414	.0950
636 000	26	2	.1564	7	1216	.990	780	.147	.147	.1618	.1618	.412	.0946
636 000	30	2	.1456	19	.0874	1.019	780	147	.147	.1618	.1618	.406	.0937
605 000	54	3	.1059	7	1059	0 953	750	.154	155	.1695	1775	.417	.0957
605 000	26	2	.1525	7	.1186	966	760	.154	154	.1700	.1720	.415	.0953
556 500	26	2	.1463	7	.1138	.927	730	.168	168	.1849	.1859	.420	.0965
556 500	30	2	.1362	7	1362	.953	730	.168	.168	.1849	.1859	.415	.0957
500 000	30	2	.1291	7	.1291	.904	690	.187	.187	.206	Same as d-c	.421	.0973
477 000	26	2	.1355	7	1054	.858	670	.196	.196	.216		.430	.0988
477 000	30	2	.1261	7	1261	.883	670	.196	.196	.216		.424	.0980
397 500	26	2	.1236	7	0961	.783	590	.235	Same as d-c	.259		.441	.1015
397 500	30	2	.1151	7	.1151	.806	600	.235		.259		.435	.1006
336 400	26	2	.1138	7	.0885	.721	530	.278		.306		.451	.1039
336 400	30	2	.1059	7	1059	.741	530	.278		.306		.445	.1032
300 000	26	2	.1074	7	0835	.680	490	.311		.342		.458	.1057
300 000	30	2	.1000	7	.1000	.700	500	.311		.342		.452	.1049
266 800	26	2	.1013	7	.0788	.642	460	.350		.385		.465	.1074

* Data courtesy of Aluminum Company of America.

† For conductor at 75°C, air at 25°C, wind 1.4 miles per hour (2 feet per second); frequency = 60 cycles.

TABLE A3-3

INDUCTIVE REACTANCE SPACING FACTOR (x_d)

Ohms per Conductor per Mile

Fundamental equations:

$$z_1 = z_2 = r_a + j(x_a + x_d) \qquad \text{and} \qquad z_0 = r_a + r_e + j(x_a + x_e - 2x_d)$$

Table gives x_d at 60 cycles; $x_d = 0.2794 \log_{10} d$; d = separation, feet

Separation

Feet	Inches 0	1	2	3	4	5	6	7	8	9	10	11
0	—	- 3015	- 2174	- 1682	- 1333	- 1062	- 0841	- 0654	- 0492	- 0349	- 0221	- 0106
1	0	0097	0187	0271	0349	0423	0492	0558	0620	0679	0735	0789
2	0841	0891	0938	0984	1028	1071	1112	1152	1190	1227	1264	1299
3	1333	1366	1399	1430	1461	1491	1520	1549	1577	1604	1631	1657
4	1682	1707	1732	1756	1779	1802	1825	1847	1869	1891	1912	1933
5	1953	1973	1993	2012	2031	2050	2069	2087	2105	2123	2140	2157
6	2174	2191	2207	2224	2240	2256	2271	2287	2302	2317	2332	2347
7	2361	2376	2390	2404	2418	2431	2445	2458	2472	2485	2498	2511
8	2523											
9	2666											
10	2794											
11	2910											
12	3015											
13	3112											
14	3202											
15	3286											
16	3364											
17	3438											
18	3507											
19	3573											
20	3635											
21	3694											
22	3751											
23	3805											
24	3856											
25	3906											
26	3953											
27	3999											
28	4043											
29	4086											
30	4127											
31	4167											
32	4205											
33	4243											
34	4279											
35	4314											
36	4348											
37	4382											
38	.4414											
39	4445											
40	4476											
41	4506											
42	4535											
43	4564											
44	4592											
45	4619											
46	4646											
47	4672											
48	4697											
49	.4722											

TABLE A3-4

ZERO-SEQUENCE RESISTANCE AND INDUCTIVE REACTANCE FACTORS (r_e, x_e)*

Ohms per Conductor per Mile

	ρ meter-ohms	Frequency 25 cycles	50 cycles	60 cycles
r_e	All	0 1192	0 2383	0 2860
x_e	1	921	1 736	2 050
	5	1 043	1 980	2 343
	10	1 095	2 085	2 469
	50	1 217	2 329	2 762
	100†	**1 270**	**2 434**	**2 888**
	500	1 392	2 679	3 181
	1000	1 444	2 784	3 307
	5000	1 566	3 028	3 600
	10 000	1 619	3 133	3 726

*From formulas: $r_e = 0.004764f$ $\quad x_e = 0.006985f \log_{10} 4{,}665{,}600 \dfrac{\rho}{f}$ where f = frequency and ρ = resistivity (meter-ohm).

† This is an average which may be used in the absence of definite information.

TABLE A3-5
SHUNT CAPACITIVE REACTANCE SPACING FACTOR (x_d')
Megohms per Conductor per Mile

Fundamental equations:

$$x_1' = x_2' = x_a' + x_d' \qquad \text{and} \qquad x_0' = x_a' + x_e' - 2x_d'$$

Table gives x_d' at 60 cycles; $x_d' = 0.06831 \log_{10} d$; d = separation, feet

Separation

Feet	Inches											
	0	1	2	3	4	5	6	7	8	9	10	11
0	—	- 0737	- 0532	- 0411	- 0326	- 0260	- 0206	-.0160	- 0120	- 0085	-.0054	- 0026
1	0	0024	.0046	0066	0085	0103	0120	0136	0152	0166	.0180	.0193
2	.0206	0218	0229	0241	.0251	0262	0272	0282	.0291	.0300	.0309	.0318
3	.0326	0334	.0342	0350	0357	0365	0372	0379	.0385	.0392	0399	.0405
4	.0411	.0417	0423	0429	0435	0441	0446	0452	0457	0462	.0467	.0473
5	.0478	0482	.0487	0492	0497	0501	0506	0510	0515	.0519	.0523	.0527
6	.0532	0536	0540	0544	0548	0552	0555	0559	.0563	.0567	.0570	.0574
7	.0577	.0581	.0584	.0588	.0591	0594	0598	0601	.0604	.0608	.0611	.0614
8	.0617											
9	.0652											
10	.0683											
11	.0711											
12	.0737											
13	.0761											
14	.0783											
15	.0803											
16	.0823											
17	.0841											
18	.0858											
19	.0874											
20	.0889											
21	.0903											
22	.0917											
23	.0930											
24	.0943											
25	.0955											
26	.0967											
27	.0978											
28	.0989											
29	.0999											
30	.1009											
31	.1019											
32	.1028											
33	.1037											
34	.1046											
35	.1055											
36	.1063											
37	.1071											
38	.1079											
39	.1087											
40	.1094											
41	.1102											
42	.1109											
43	.1116											
44	.1123											
45	.1129											
46	.1136											
47	.1142											
48	.1149											
49	.1155											

TABLE A3-6
ZERO-SEQUENCE SHUNT CAPACITIVE REACTANCE FACTOR (x_e')*
Megohms per Conductor per Mile

Conductor height above ground, feet	Frequency		
	25 cycles	50 cycles	60 cycles
10	0.640	0 320	0 267
15	.727	.363	.303
20	.788	.394	.328
25	.836	.418	.348
30	.875	.437	.364
40	.936	.468	.390
50	.984	.492	.410
60	1.023	.511	.426
70	1.056	.528	.440
80	1.084	.542	.452
90	1.109	.555	.462
100	1.132	.566	.472

* From formula: $x_e' = \frac{12.30}{f} \log_{10} 2\,h$
where h = height above ground and f = frequency.

APPENDIX 4

REFERENCES

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